• Logrithmic Differentiation

$y = f(x)$

(1) $\ln y = \ln f(x) = \cdots$, (2) implicit differentiation, (3) solve the equation for y'

Ex. 7 $\quad y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

$$(\ln y(x))' = \left[\ln x^{\frac{3}{4}} + \ln (x^2+1)^{\frac{1}{2}} - \ln (3x+2)^5\right]'$$

$$= \left[\frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)\right]'$$

$$\frac{1}{y(x)}\cdot y'(x) = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\,\frac{1}{x^2+1}\cdot 2x - 5\,\frac{3}{3x+2} = \frac{3}{4x} + \frac{\cancel{2}x}{\cancel{2}(x^2+1)} - \frac{15}{3x+2}$$

$$y'(x) = \left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]\cdot y$$

$y = x^r \longrightarrow y' = r x^{r-1}$

$y = b^x \longrightarrow \underline{y' = b^x \ln b}$

$b^x = e^{\ln b^x} = e^{x\ln b}$

$$\left(e^{x\ln b}\right)' = e^{x\ln b}\cdot \underline{\underline{\ln b}}$$

Ex. 8 $\quad y = x^{\sqrt{x}}$

Solution 1 $\quad (\ln y(x))' = \left(\ln x^{\sqrt{x}}\right)' = \left(\sqrt{x}\,\ln x\right)'$

$$\frac{1}{y(x)}\cdot y'(x) = \frac{1}{2}x^{-\frac{1}{2}}\ln x + \sqrt{x}\cdot\frac{1}{x} = \frac{1}{2}x^{-\frac{1}{2}}\ln x + x^{-\frac{1}{2}} = \left(\frac{1}{2}\ln x + 1\right)\Big/\sqrt{x}$$

$$y' = \frac{\ln x + 2}{2\sqrt{x}}\cdot x^{\sqrt{x}}$$

Solution 2 $\quad y' = \left(x^{\sqrt{x}}\right)' = \left(e^{\ln x^{\sqrt{x}}}\right)'$

$$= \left(e^{\sqrt{x}\ln x}\right)' = e^{\sqrt{x}\ln x}\cdot\frac{\ln x + 2}{\sqrt{x}} = \frac{\ln x + 2}{\sqrt{x}}\, x^{\sqrt{x}}$$

§3.7 Rates of Change in the Natural and Social Sciences

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

- Physics: $s = f(t)$ — the position function of a moving particle in a straight line

velocity $v(t) = \frac{ds}{dt}$, acceleration $a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$

Ex. 1 The position of a particle is given by $s = f(t) = t^3 - 6t^2 + 9t$ $\left\langle \begin{array}{l} s - \text{meters} \\ t - \text{seconds} \end{array} \right.$

(a) Find the velocity at time t. $v(t) = s'(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3)$
$= 3(t-1)(t-3)$

(b) What is the velocity after 2 s? After 4 s? $v(2) = 3 \cdot 1 \cdot (-1) = -3$, $v(4) = 3 \cdot 3 \cdot 1 = 9$

(c) When is the particle at rest? $0 = v(t) = 3(t-1)(t-3) \Longrightarrow t = 1$ or $t = 3$

(d) When is the particle moving forward (in the positive direction)?

+   −   +

$t=0$ $t=1$ $t=3$

(e) Draw a diagram to represent the motion of the particle.

(f) Find the total distance traveled by the particle during the first 5 s.

(g) Find the acceleration at time t and after 4 s. $a(t) = v'(t) = 3(2t - 4)$

(h) Graph the position, velocity, and acceleration functions for $0 \le t \le 5$.

(i) When is the particle speeding up? When is it slowing down?

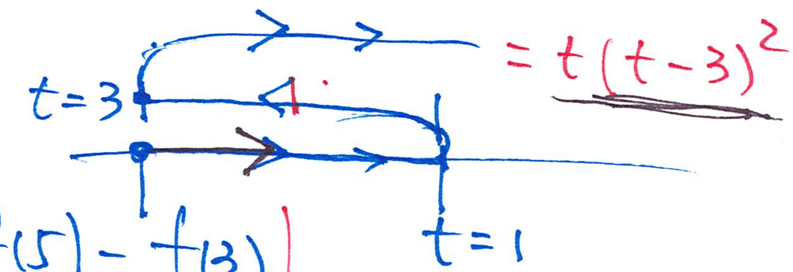
$v>0 \quad v \nearrow \Leftrightarrow a>0$

$v<0 \quad v \searrow \Leftrightarrow a<0$

$f(t) = t^3 - 6t^2 + 9t = t\left[t^2 - 6t + 9\right] = \underline{t(t-3)^2}$

(e) $+$

$t=0 \qquad t=1 \qquad t=3$



$d = |f(1) - f(0)| + |f(3) - f(1)| + |f(5) - f(3)|$

$= (1 - 6 + 9) + |(3^3 \underbrace{- 6\cdot 3^2 + 9\cdot 3}_{= 18}) - \underline{4}| + |(5^3 - 6\cdot 5^2 + 9\cdot 5) - 18|$

$-25 + 45$

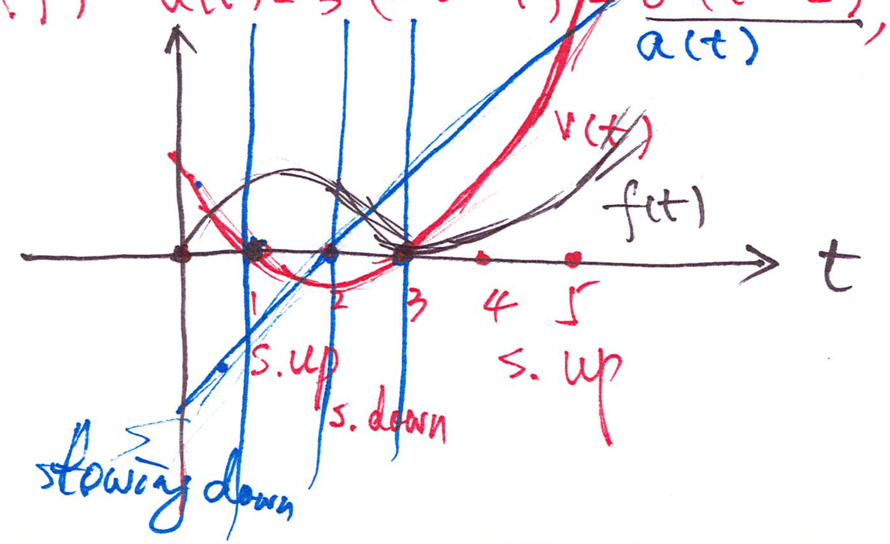
$= 4 + \underline{14} + 2 = 38$

(f) $a(t) = 3(2t-4) = \underline{6(t-2)}_{a(t)}$, $a(4) = 6 \cdot 2 = 12$.

$v(t) = 3(t-1)(t-3)$

$v(5) = 3 \cdot 4 \cdot 2 = 24$

$a(5) = 6 \cdot 3 = 18$



§3.9 Related Rates (2 lectures)

Ex. 1 Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

relation

$$V(t) = \frac{4}{3}\pi r(t)^3$$

Given: the rate of increase of the volume is 100 cm^3/s

V — volume

$$\frac{dV}{dt} = 100\ cm^3/s$$

Unknown: the rate of increase of the radius when the diameter is 50 cm

r — radius

$$\left.\frac{dr}{dt}\right|_{r=25} = ?$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cancel{3} \cdot r^2 \cdot \frac{dr}{dt}$$

$$\left.\frac{dr}{dt}\right|_{r=25} = \frac{1}{4\pi r^2} \cdot \left.\frac{dV}{dt}\right|_{r=25}$$

$$= \frac{1}{4\pi \cdot 25^2} \cdot 100 = \frac{1}{25\pi}\ cm/s$$