#14 If a snow ball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

$$\frac{dS}{dt} = 1 \ cm^2/min \qquad \left.\frac{d\ell}{dt}\right|_{\ell=10} = ?$$

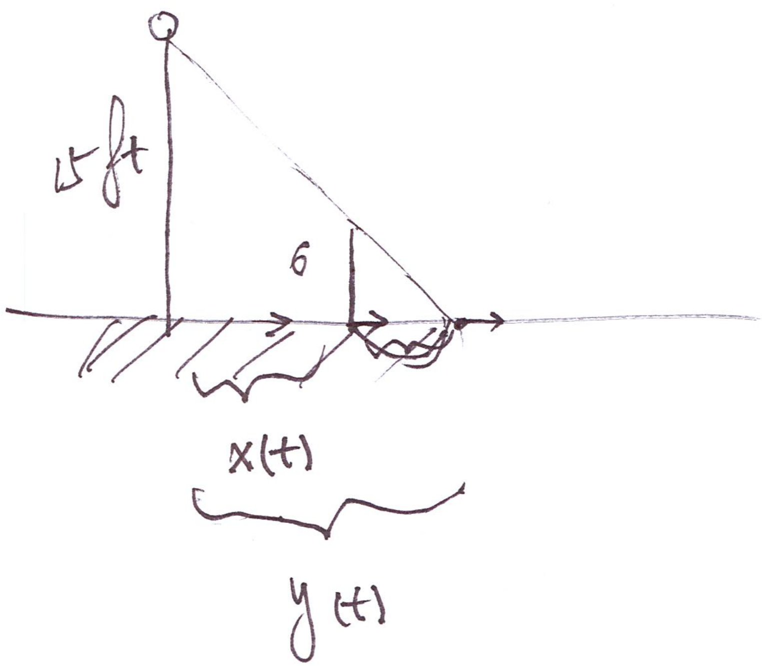
$$S = 4\pi r^2 = \pi (2r)^2 = \pi \ell^2$$

$$\frac{dS}{dt} = \pi \cdot 2\ell \frac{d\ell}{dt} \Rightarrow \left.\frac{d\ell}{dt}\right|_{\ell=10} = \left.\frac{1}{2\pi\ell}\frac{dS}{dt}\right|_{\ell=10}$$

$$= \frac{1}{20\pi} \cdot 1 = \frac{1}{20\pi} \ cm/min$$

#15 A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

$$\frac{dx}{dt} = 5 \text{ ft/s}, \qquad \left.\frac{dy}{dt}\right|_{x=40} = ?$$



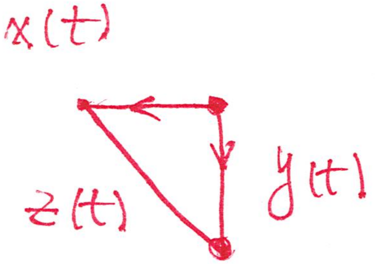
$$\frac{6}{15} = \frac{y-x}{y}$$

$$\left(1 - \frac{6}{15}\right) y = x$$

$$y = \frac{5}{3} x$$

$$\frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt} = \frac{5}{3} \cdot 5 = \frac{25}{3}$$

#17 Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?



$$\frac{dy}{dt} = 60 \text{ mi/h}, \quad \frac{dx}{dt} = 25 \text{ mi/h}$$

$$\left.\frac{dz}{dt}\right|_{t=2} = ?$$

$$z^2 = x^2 + y^2$$

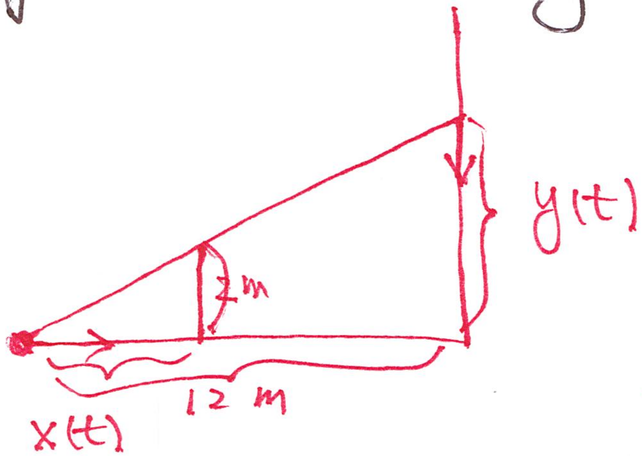
$$2z \cdot z' = 2x \cdot x' + 2y \cdot y' \Rightarrow \left. z' \right|_{t=2} = \left. \frac{x x' + y y'}{z} \right|_{t=2}$$

$x(2) = \cancel{\ } \cdot 2 \cdot 25 = 50$ ~~mm~~ mi

$y(2) = 2 \cdot 60 = 120$ mi

$z(2) = \sqrt{x(2)^2 + y(2)^2} = \sqrt{}$

#18 A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?



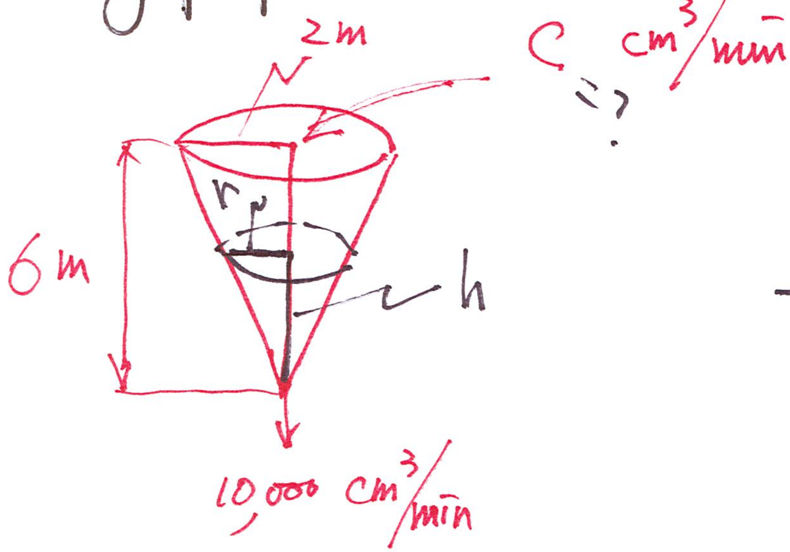
$$\frac{dx}{dt} = 1.6 \text{ m/s}, \quad \left.\frac{dy}{dt}\right|_{x=8} = ?$$

$$\frac{x}{12} = \cancel{\frac{2}{y}} \quad \frac{2}{y}$$

$$y = \frac{24}{x} = 24x^{-1}, \quad \left. y' \right|_{x=8} = \left. -24x^{-2}x' \right|_{x=8} = \left. -\frac{24}{x^2}\cdot x' \right|_{x=8}$$

$$= -24\,\frac{1}{8^2}\cdot 1.6$$

#25 Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same ~~the~~ time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the water height is 2m, find the rate at which water is being pumped into the tank.



$$\left.\frac{dh}{dt}\right|_{h=2m} = 20 \text{ cm/min}$$

$$\frac{dV}{dt} = C - 10,000 \quad cm^3/min$$

$$V = \cancel{\frac{4}{3}\pi}\ \frac{1}{3}\pi r^2 h.$$

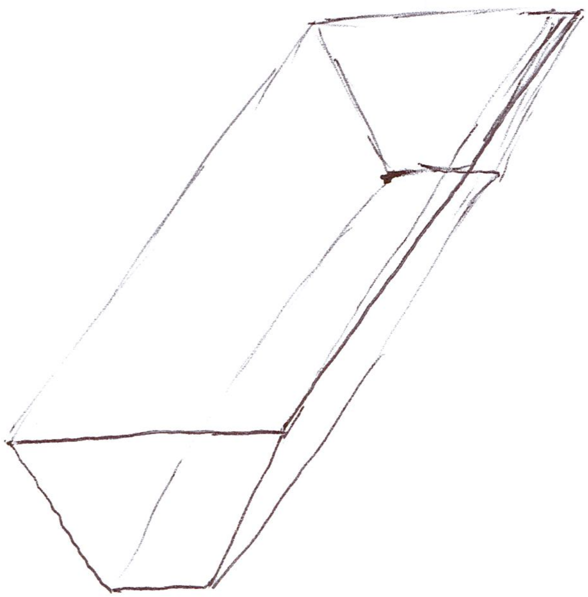
$$\frac{r}{2} = \frac{h}{6}$$

$$= \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{1}{3^3}\pi h^3$$

$$\Rightarrow r = \frac{1}{3}h \qquad C - 10,000 = \frac{dV}{dt} = \frac{1}{9}\pi h^2 \cdot \frac{dh}{dt} = \frac{\pi}{9}\cdot 2^2 \cdot 20$$

#27 A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water ~~with~~ at the rate of $0.2\ m^3/min$, how fast is the water level rising when the water is 30 cm deep?

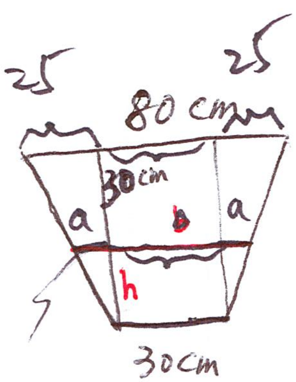
$$\frac{dV}{dt} = 0.2\ m^3/min = 100^3 \cdot 0.2$$



$$\left.\frac{dh}{dt}\right|_{h=30\,cm} = ?$$

$$V = 10 \cdot A = 10 \cdot \frac{30 + (30 + 2a)}{2} h$$

$$\frac{a}{25} = \frac{h}{50} \Rightarrow a = \frac{h}{2}$$



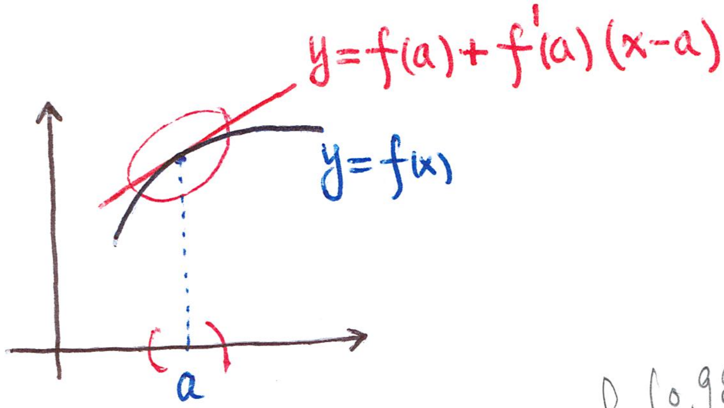
$$A = \frac{30+b}{2} h$$

0.

$$V = 5 \cdot (60 + h) h = 5h^2 + 300h$$

$$= \left.\frac{dV}{dt}\right|_{h=30} = (10h + 300) \left.\frac{dh}{dt}\right|_{h=30} = 600 \cdot \frac{dh}{dt}$$

§3.10 Linear Approximations and Differentials



when x near a

$$f(x) \approx \underbrace{L(x)}_{\text{Linearization of } f \text{ at } a} \equiv f(a) + f'(a)(x-a)$$

Linearization of f at a.

Linear approximation of f at a

$f(0.98)$

Ex. 1 Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a=1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations over- or under-estimates?

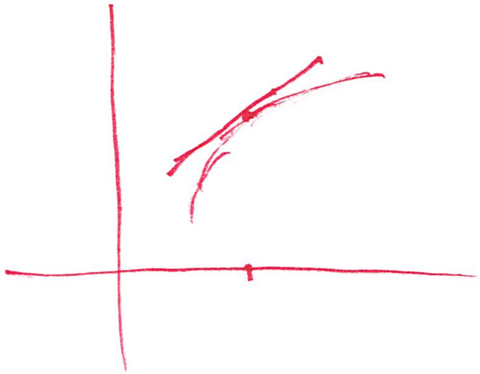
$$f' = \frac{1}{2}(x+3)^{-\frac{1}{2}} \cdot 1 = \frac{1}{2\sqrt{x+3}}$$

$$L(x) = f(1) + f'(1)(x-1) = \sqrt{1+3} + \frac{1}{2\sqrt{1+3}} \cdot (x-1)$$

$$= 2 + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{7}{4}$$

$$f(0.98) = \sqrt{3.98} = \sqrt{0.98+3} \approx 2 + \frac{1}{4}(0.98-1) = 2 + \frac{1}{4}(-0.02) =$$

$$f(1.05) = \sqrt{4.05} = \sqrt{1.05+3} \approx 2 + \frac{1}{4}(1.05-1) = 2 + \frac{1}{4}\cdot(0.05) =$$



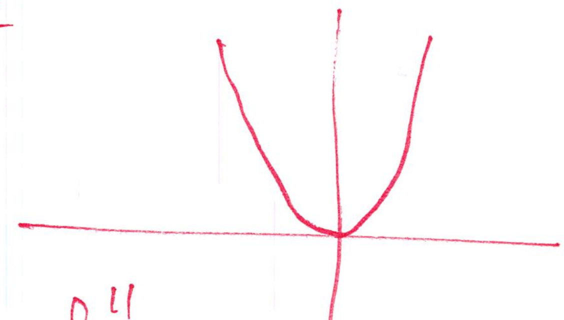
$f(x) = \sqrt{x+3}$

$f'(x) = \frac{1}{2}(x+3)^{-\frac{1}{2}} > 0$

$f''(x) = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)(x+3)^{-\frac{3}{2}}$

$= -\frac{1}{4}\cdot\frac{1}{(x+3)^{3/2}} < 0$ for x is close to 1

$y = f(x) = x^2$

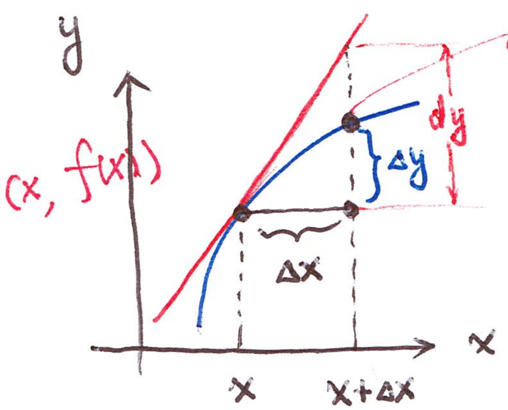


$f''(x) = 2 > 0$

- <u>differentials</u> $\quad y = f(x)$, $f(x)$ is differentiable

<u>differential</u> dx is an independent variable

<u>differential</u> $\boxed{dy = f'(x)\,dx}$ is a dependent variable



let $dx = \Delta x$

$$\Longrightarrow \quad \Delta y = f(x+\Delta x) - f(x) \approx f'(x)\Delta x = f'(x)dx = dy$$

<u>Ex. 3</u> $y = f(x) = x^3 + x^2 - 2x + 1$. Compare the values of Δy and dy when x changes

(a) from 2 to 2.05

(b) from 2 to 2.01