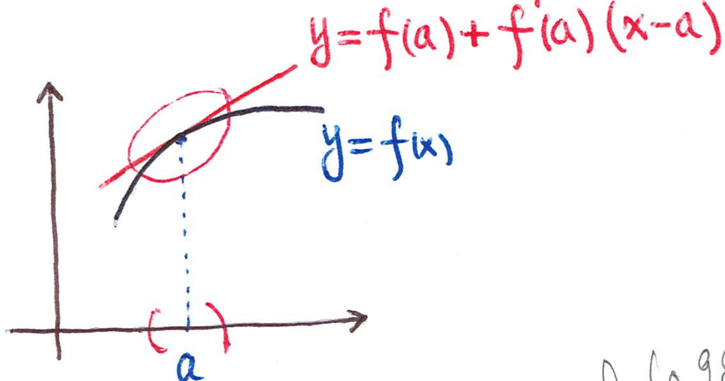
§3.10 Linear Approximations and Differentials



when x near a

$$f(x) \approx L(x) \equiv f(a)+f'(a)(x-a)$$

Linearization of f at a.

Linear approximation of f at a

Ex. 1 Find the linearization of the function $f(x)=\sqrt{x+3}$ at $a=1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations over- or under-estimates?

$f(0.98)$

$$f' = \frac{1}{2}(x+3)^{-\frac{1}{2}} \cdot 1 = \frac{1}{2\sqrt{x+3}}$$

$$L(x) = f(1) + f'(1)(x-1) = \sqrt{1+3} + \frac{1}{2\sqrt{1+3}} \cdot (x-1)$$

$$= 2 + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{7}{4}$$

$$f(0.98) = \sqrt{3.98} = \sqrt{0.98+3} \approx 2 + \frac{1}{4}(0.98-1) = 2 + \frac{1}{4}(-0.02) =$$

$$f(1.05) = \sqrt{4.05} = \sqrt{1.05+3} \approx 2 + \frac{1}{4}(1.05-1) = 2 + \frac{1}{4} \cdot (0.05) =$$

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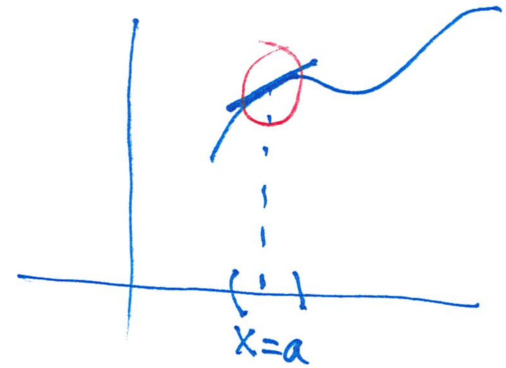
$$f(x) \approx f(a) + f'(a)(x-a)$$



$$f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x$$

$$\boxed{f(x+\Delta x) - f(x)} \approx f'(x)\underbrace{\Delta x}_{dx} = dy$$

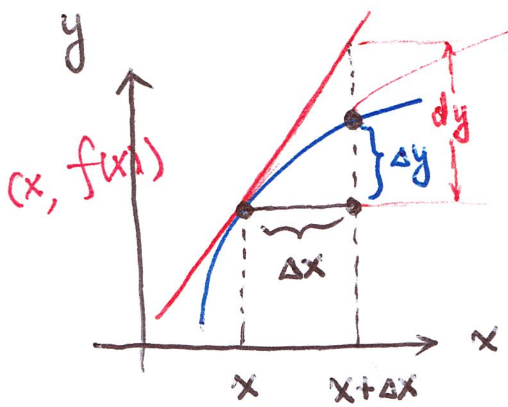
- <u>differentials</u> $\quad y=f(x)$, $f(x)$ is differentiable

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

<u>differential</u> dx is an independent variable

<u>differential</u> $\boxed{dy=f'(x)\,dx}$ is a dependent variable

$(x+\Delta x,\ f(x+\Delta x))$



let $dx=\Delta x$

$$\Longrightarrow \quad \Delta y = f(x+\Delta x)-f(x) \approx f'(x)\Delta x = f'(x)\,dx = dy$$

<u>Ex. 3</u> $y=f(x)=x^3+x^2-2x+1$. Compare the values of Δy and dy when x changes

(a) from 2 to 2.05

(b) from 2 to 2.01

(a) $\Delta x = dx = 2.05-2=0.05$

$$f(2.05)-f(2)=\left[(2.05)^3+(2.05)^2-2\cdot(2.05)+1\right]-\left[2^3+2^2-2\cdot 2+1\right]$$

$$= 9.717625-9=0.717625$$

$$dy=f'(2)\Delta x=\left[3x^2+2x-2\right]_{x=2}\cdot 0.05 = 14\times 0.05=0.7$$

$$\Delta y - dy = 0.017625$$

Examples

$dy = f'(x)\,dx = f'(x)\,\Delta x$

#14 Find the differential of $y = e^{\tan(\pi t)}$

$$dy = e^{\tan(\pi t)} \cdot \sec^2(\pi t) \cdot \pi \, dx$$

#15 Find the differential dy and evaluate dy for the given values of x and dx

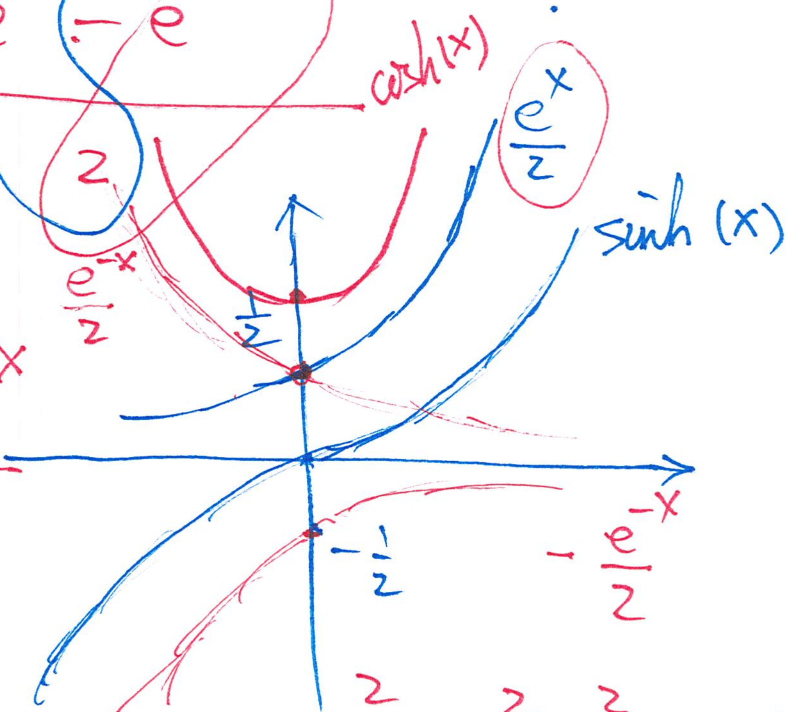
$y = e^{\frac{x}{10}}$, $x = 0$, $dx = 0.1$

$$dy = e^{\frac{x}{10}} \cdot \frac{1}{10} \cdot dx$$

$$dy\Big|_{x=0} = \frac{1}{10} \cdot e^{\frac{0}{10}} \cdot 0.1 = \frac{1}{10} \cdot 0.1 = 0.01$$

§3.11 Hyperbolic Functions

hyperbolic sine $\qquad \sinh(x) = \dfrac{e^x - e^{-x}}{2}$



hyperbolic cosine $\qquad \cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\|$$

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= e^x \cdot e^{-x} = 1$$

$$(a+b)^2 = a^2 + b^2 + \cancel{2ab}$$

$$-)\ (a-b)^2 = a^2 + b^2 - \cancel{2ab}$$

$$(a+b)^2 - (a-b)^2 = 2(a^2 + b^2)$$

Ex. 1 Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \text{sech}^2 x$.

Proof (a) $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \left[\frac{e^x + e^{-x}}{\cancel{2}} + \frac{e^x - e^{-x}}{2}\right]\left[\frac{e^x + e^{-x}}{\cancel{2}} - \frac{(e^x - e^{-x})}{2}\right]$$

$$= e^x \cdot e^{-x} = 1$$

(b) $1 - \tanh^2 x = 1 - \left(\frac{\sinh x}{\cosh x}\right)^2 = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

- derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\left(e^x - e^{-x}\cdot\underline{(-1)}\right)$$

$$= \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh x$$

$$\underline{\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x} \qquad \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Ex. 2 $$\frac{d}{dx}(\cosh\sqrt{x}) = \sinh\left(\sqrt{x}\right)\cdot\frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}\sinh(\sqrt{x})$$

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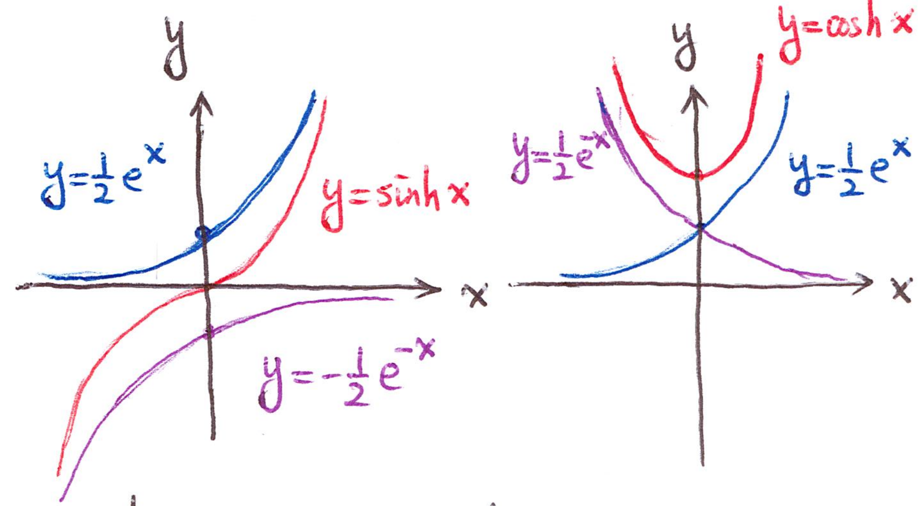

- hyperbolic sine $\quad \sinh x = \dfrac{e^x - e^{-x}}{2}$

- hyperbolic cosine $\quad \cosh x = \dfrac{e^x + e^{-x}}{2}$

- $\tanh = \dfrac{\sinh x}{\cosh x}$, $\coth x = \dfrac{\cosh x}{\sinh x}$, $\operatorname{sech} x = \dfrac{1}{\cosh x}$, $\operatorname{csch} x = \dfrac{1}{\sinh x}$

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{-(e^x - e^{-x})}{2} = -\sinh(x)$$

- hyperbolic identities

$\sinh(-x) = -\sinh x$ (odd function)

$\cosh(-x) = \cosh x$ (even function)

$\cosh^2 x - \sinh^2 x = 1 \quad \left(\dfrac{e^x+e^{-x}}{2}\right)^2 - \left(\dfrac{e^x-e^{-x}}{2}\right)^2$

$1 - \tanh^2 x = \operatorname{sech}^2 x \quad = \left(\dfrac{e^x}{2}\right)^2 + \left(\dfrac{e^{-x}}{2}\right)^2$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$\sin(-x) = \cancel{} -\sin(x)$

$\cos(-x) = \cos x$

$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$