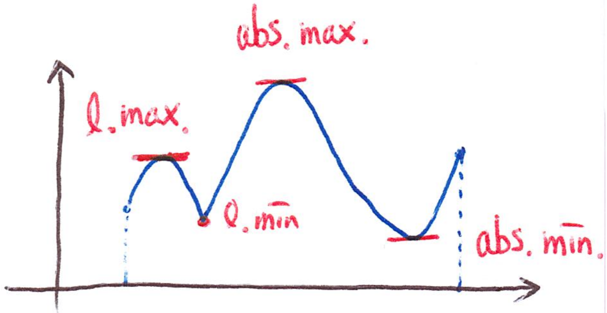
Chapter 4. Application of Differentiation (9 lectures)

§4.1 Maximum and Minimum Values



Def. (absolute max./min) $c \in \mathrm{dom}(f)$

(1) $f(c)$ is the absolute maximum value of f on D

$$\iff f(c) \geq f(x) \quad \forall x \in D$$

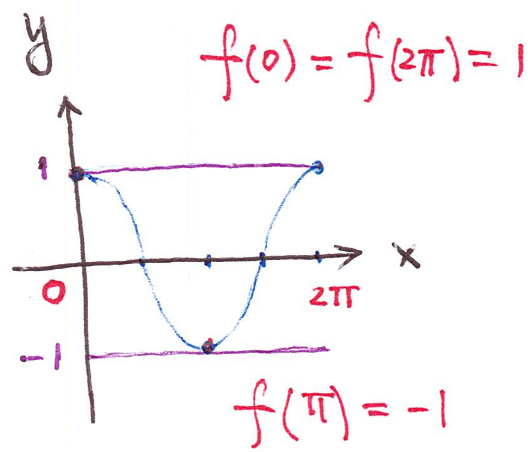
(2) $f(c)$ is the absolute minimum value of f on D

$$\iff f(c) \leq f(x) \quad \forall x \in D$$

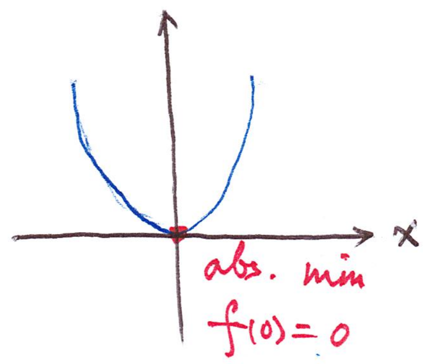
Def. (local max./min.)

(1) $f(c)$ is a local maximum value of $f \iff f(c) \geq f(x)$ when x is near c.

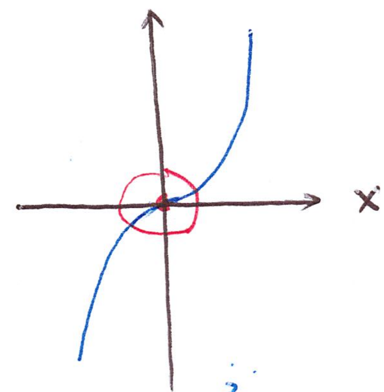
(2) $f(c)$ is a local minimum value of $f \iff f(c) \leq f(x)$ when x is near c



$y = \cos x$



$y = x^2$

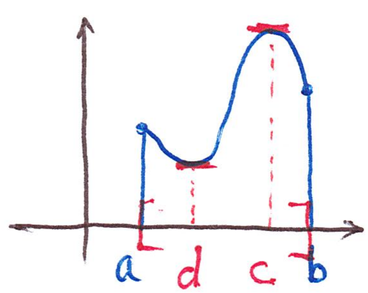


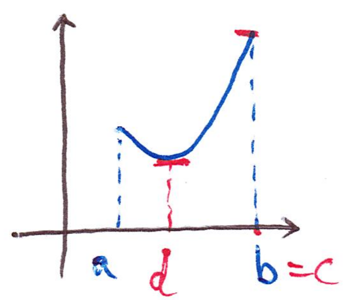
$y = x^3$

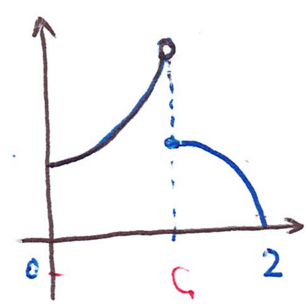
The Extreme Value Thrm

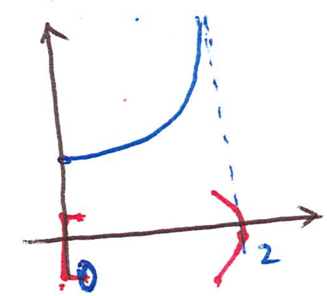
Assume that f is continuous on a closed interval $[a, b]$

$\Longrightarrow$ f attains an absolute max. value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$





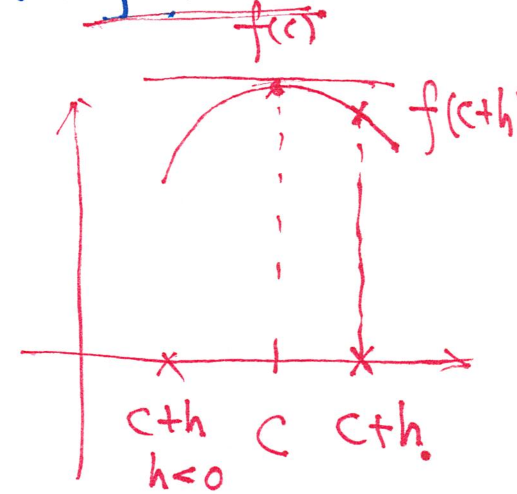


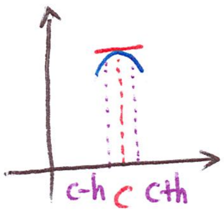


Fermat's Theorem

Assume that f has a local maximum or minimum at c and $f'(c)$ exists

$$\Longrightarrow \boxed{f'(c)=0}$$





$$\frac{f(c+h)-f(c)}{h} \begin{cases} \leq 0 & h>0 \\ \geq 0 & h<0 \end{cases}$$

$$\frac{f(c+h)-f(c)}{h} \leq 0.$$

$$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$$

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0$$

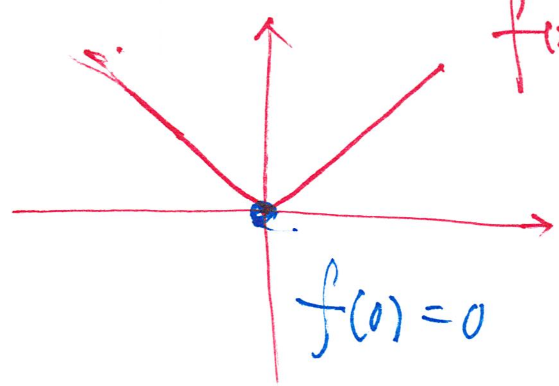
Ex. 5 $f(x)=x^3$

$f'(x)=3x^2=0 \Rightarrow x=0$

$f(0)=0$

Ex. 6 $f(x)=|x|$

$$= \begin{cases} x & x\geq 0 \\ -x & x\leq 0 \end{cases}$$



Def. (critical number)

$c \in \mathrm{dom}(f)$ is a critical number $\iff$ either $f'(c)=0$ or $f'(c)$ does not exist.

Ex. 7 Find the critical numbers of $f(x) = x^{\frac{3}{5}}(4-x)$. $= 4x^{\frac{3}{5}} - x^{\frac{8}{5}}$

$$f'(x) = 4\cdot\frac{3}{5}\cdot x^{-\frac{2}{5}} - \frac{8}{5}x^{\frac{3}{5}} = \frac{4}{5}x^{-\frac{2}{5}}\left[3-2x\right] = 0 \implies 3 = 2x$$

$$\boxed{x = \frac{3}{2}}$$

$x=0$ $\quad f'(0)$ DNE. $\qquad f\left(\frac{3}{2}\right)=0$

Ex. 8 Find the absolute extreme values of $f(x) = x^3 - 3x^2 + 1$ on $\left[-\frac{1}{2}, 4\right]$.

① $0 = f'(x) = 3x^2 - 6x = 3x(x-2) \implies x=0$ or $x-2=0$

$\implies x=0$ or $x=2$

$f(0)=1$, $f(2) = 2^3 - 3\cdot 2^2 + 1 = 8 - 12 + 1 = -3$ abs. min

loc. max. $\qquad$ loc. min

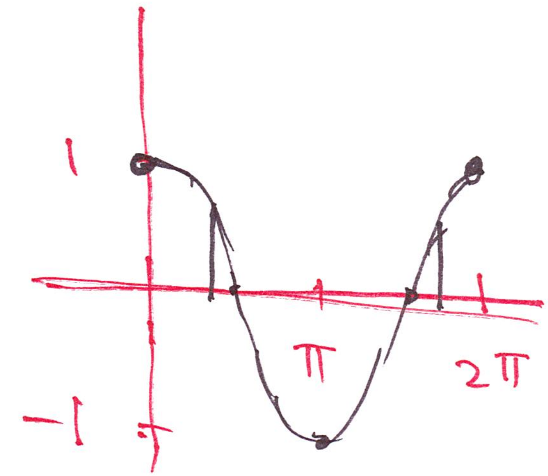
② $f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 3\cdot\left(-\frac{1}{2}\right)^2 + 1 = -\frac{1}{8} - \frac{3}{4} + 1 = \frac{-1-6+8}{8} = \frac{1}{8}$

$f(4) = 4^3 - 3\cdot 4^2 + 1 = 16 + 1 = 17$

abs. max.

Ex. 9 Find the absolute extreme value of $f(x) = x - 2\sin x$ on $[0, 2\pi]$.

① $f'(x) = 1 - 2\cos x = 0 \Rightarrow \cos x = \frac{1}{2}$



$x = \frac{\pi}{3}, \quad \frac{5\pi}{3}$

$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - 2\cdot\frac{\sqrt{3}}{2}$

$= \frac{\pi}{3} - \sqrt{3} < 0$ abs. min

$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{3} + \sqrt{3}$

abs. max.

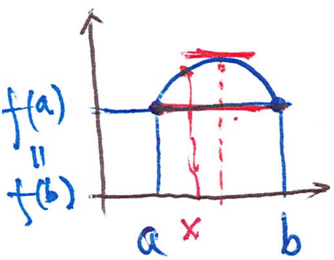
② $f(0) = 0 - 0 = 0$

$f(2\pi) = 2\pi$

§4.2 The Mean Value Theorem

Rolle's Theorem

Assume that
$$\left\{\begin{array}{l}(1)\ f \text{ is continuous on } [a,b] \\ (2)\ f \text{ is differentiable on } (a,b) \\ (3)\ f(a)=f(b)\end{array}\right. \Longrightarrow \exists\, c \in (a,b) \text{ such that } f'(c)=0$$



Proof

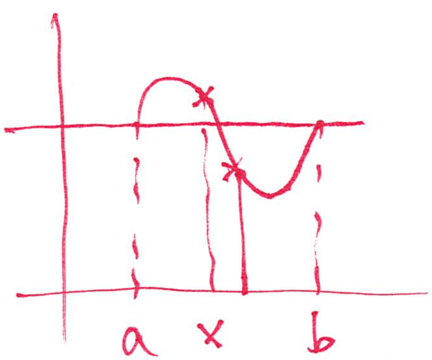
Case (a) $f(x)=k$ constant $\Longrightarrow f'(x)=0$

Case (b) For some $x\in(a,b)$, $f(x)>f(a)=f(b)$

f is continuous on $[a,b] \Longrightarrow \exists\, c\in[a,b]$ s.t. $f(c)$ is the abs. max. $\Rightarrow f'(c)=0$

$\Longrightarrow f(c)\geq f(x)>f(a)=f(b)$

$\Longrightarrow c\neq a$ or $b \Longrightarrow c\in(a,b)$



Case (c) For some $x\in(a,b)$, $f(x)<f(a)$

Ex. 1 A ball is thrown directly upward.

$s = f(t)$ — position of a moving object.

$f(a) = f(b) \implies \exists\, c \in (a,b)$ s.t. $f'(c) = 0$ — velocity.

Ex. 2 Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

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$\qquad\qquad\qquad\qquad\qquad\qquad f(x)$

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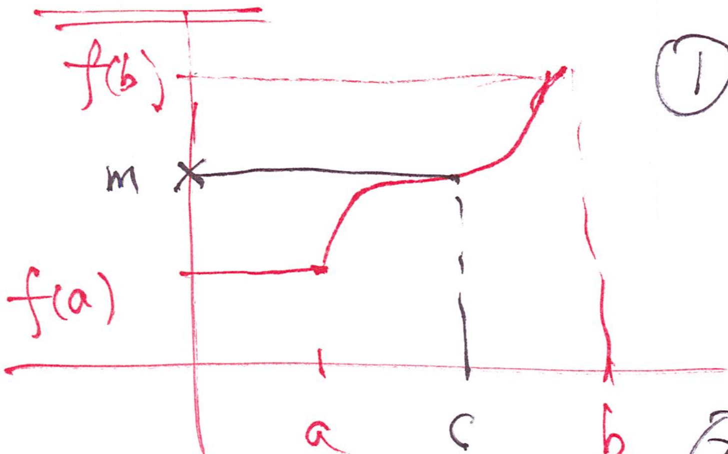
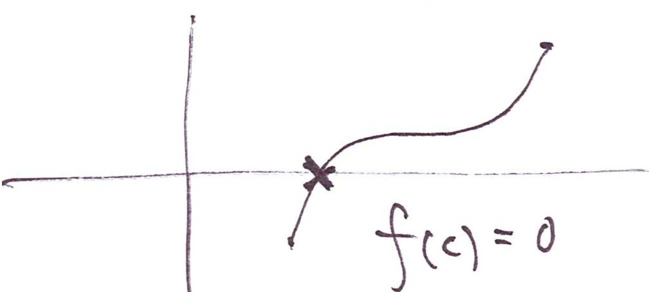


$f(c) = m$



① $f(0) = -1 < 0$

$f(1) = 1 > 0$

$\exists\, c \in (0,1)$ s.t. $f(c) = 0$

t_1, t_2

$s'(c) = 0$

② $f'(x) = 3x^2 + 1 \geq 1 > 0$

Assume that it has at least 2 solutions:

$\exists\, c_1$, and c_2 s.t. $f(c_1) = 0 = f(c_2) = 0$

$\Rightarrow \exists\, c$ s.t. $f'(c) = 0$

$f'(x) = 3x^2 + 1 > 0$