Ex. 7 Sketch the graph of the function $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$.

$$f'(x) = \frac{2}{3}(x)^{-\frac{1}{3}}(6-x)^{\frac{1}{3}} + x^{\frac{2}{3}}\cdot\frac{1}{3}(6-x)^{-\frac{2}{3}}\cdot(-1) = \frac{2(6-x) - x}{\cancel{3}x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}$$

$$2(6-x) - x = 12-3x = \cancel{3}(4-x)$$

$$\begin{array}{c|ccccccc} f'(x) & - & \pm\infty & + & 0 & - & \infty & - \\ \hline & & 0 & & 4 & & 6 & \end{array}$$

loc. max. $f(4) = 4^{\frac{2}{3}}\cdot 2^{\frac{1}{3}} = 2^{\frac{4}{3}}\cdot 2^{\frac{1}{3}} = 2^{\frac{5}{3}}$

$$f''(x) = \frac{-x^{\frac{1}{3}}(6-x)^{\frac{2}{3}} - (4-x)\left[\frac{1}{3}x^{-\frac{2}{3}}(6-x)^{\frac{2}{3}} + x^{\frac{1}{3}}\cdot\frac{2}{3}(6-x)^{-\frac{1}{3}}\cdot(-1)\right]}{x^{\frac{2}{3}}\cdot(6-x)^{\frac{4}{3}}}$$

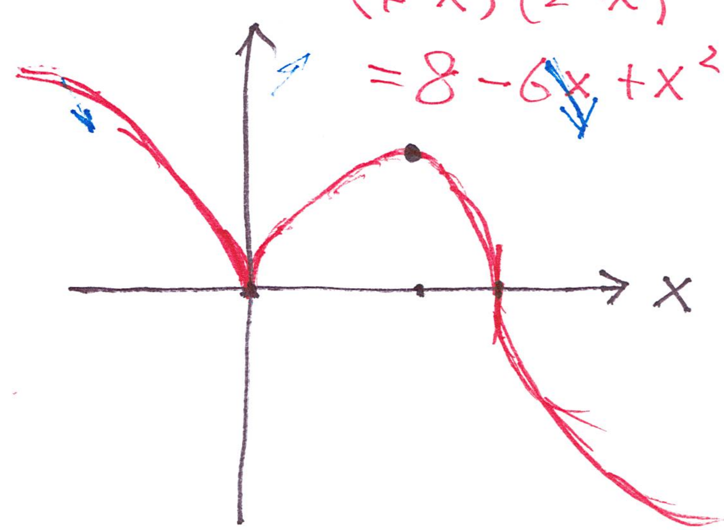
$$\frac{(6-x) - 2x}{\cancel{3}x^{\frac{2}{3}}\cdot(6-x)^{\frac{1}{3}}} \qquad (6-x)-2x = \cancel{3}(2-x)$$

$$= \frac{-x^{\frac{1}{3}}(6-x)^{\frac{2}{3}} - \dfrac{8-6x+x^2}{x^{2/3}(6-x)^{1/3}}}{x^{\frac{2}{3}}(6-x)^{\frac{4}{3}}}$$

$$(4-x)(2-x) = 8-6x+x^2$$

$$= \frac{-x(6-x) - (8-6x+x^2)}{x^{\frac{4}{3}}(6-x)^{5/3}} \qquad -x(6-x) - (8-6x+x^2) = -8$$

$$\begin{array}{c|ccccc} f''(x) & - &) & - & | & + \\ \hline & \frown & 0 & \frown & 6 & \smile \end{array}$$



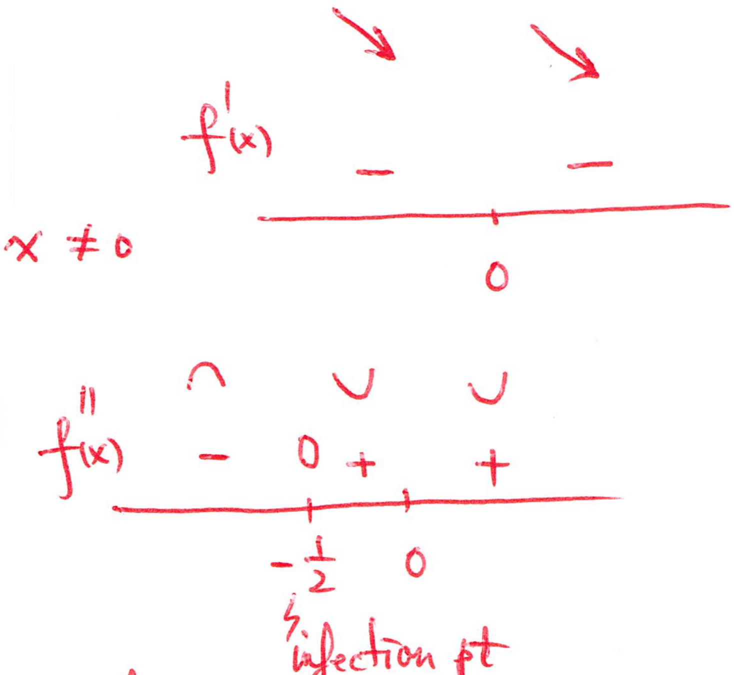
Ex. 8 Use the 1^{st} and 2^{nd} derivatives of $f(x) = e^{\frac{1}{x}}$, together with asymptotes, to sketch its graph.

$$\mathrm{dom}(f(x)) = \{x \in \mathbb{R} \mid x \neq 0\}$$

$$f'(x) = e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) = -\frac{e^{\frac{1}{x}}}{x^2} < 0 \quad \text{for all } x \neq 0$$

$$f''(x) = \left(-x^{-2} e^{\frac{1}{x}}\right)' = -\left[(-2)x^{-3} e^{\frac{1}{x}} + x^{-2} \cdot \left(-\frac{e^{\frac{1}{x}}}{x^2}\right)\right]$$

$$= e^{\frac{1}{x}} \left[\frac{2}{x^3} + \frac{1}{x^4}\right] = \frac{e^{\frac{1}{x}}}{x^4}(2x+1)$$

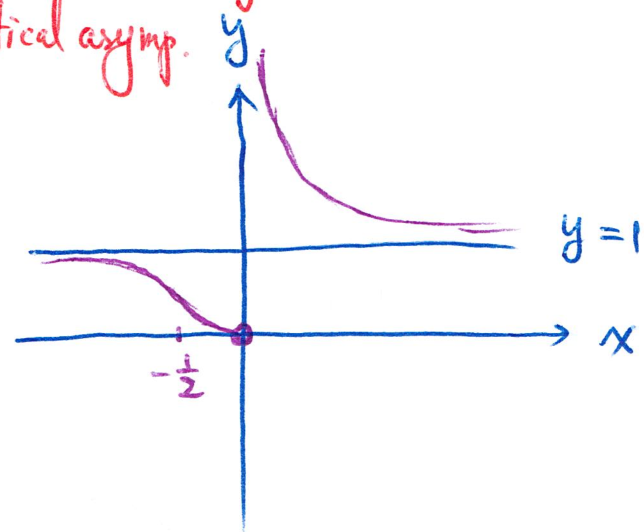


asymptotes

$$\lim_{x \to 0^+} e^{\frac{1}{x}} = \lim_{t \to \infty} e^{t} = \infty \Rightarrow x = 0 \text{ vertical asymp.}$$

$$\lim_{x \to 0^-} e^{\frac{1}{x}} = \lim_{t \to \infty} e^{-t} = 0$$

$$\lim_{x \to \pm\infty} e^{\frac{1}{x}} = e^0 = 1 \Rightarrow y = 1 \text{ horizontal asymp}$$



§4.4 Indeterminate Forms and l'Hospital's Rule

$$a^2 - b^2 = (a+b)(a-b)$$

- indeterminate forms

$$\lim_{x \to a} \frac{f(x)}{g(x)} : \quad \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

examples

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \frac{1}{2}$$

$$\lim_{x \to 1} \frac{\ln x}{x - 1} =$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

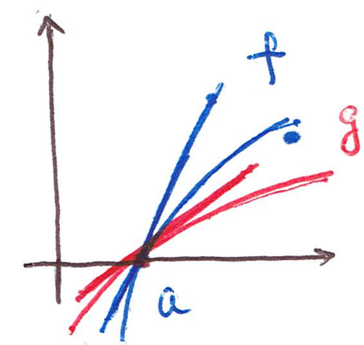
$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \frac{1}{2}$$

L'Hospital's Rule

Assume that f and g are differentiable and $g'(x) \neq 0$ on an open interval $I \ni a$ and that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

$$\Longrightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$



Proof of L'Hospital's Rule in the special case that $f(a)=g(a)=0$, f' and g' are cont, $g'(a)\neq 0$

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} = \lim \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{\cancel{x-a}}} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a}\frac{f(x)}{g(x)}$$

Examples

(1) $$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{1}{x}}{1} = \lim_{x\to 1}\frac{1}{x} = 1$$

$\ln x \to 0$, $x-1 \to 0$

(2) $$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2} = +\infty$$

$e^x \to \infty$, $x^2 \to \infty$, $x \to \infty$

$$\lim_{x\to\infty}\frac{e^x}{x^{100}} = \infty$$

$e^x \quad x^{100} \quad x^2, \quad x \quad \ln x$

(3) $$\lim_{x\to\infty} \frac{\ln x \to \infty}{x^{\frac{1}{3}} \to \infty} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-\frac{2}{3}}} = \lim_{x\to\infty} \frac{3}{x^{\frac{1}{3}}} = 0$$

$$\frac{1}{\cos^2 x} - 1 = \frac{\sin^2 x}{\cos^2 x}$$

(4) $$\lim_{x\to 0} \frac{\tan x - x \to 0}{x^3 \to 0} = \lim_{x\to 0} \frac{\sec^2 x - 1 \to 0}{3x^2 \to 0} = \lim_{x\to 0} \frac{\sin^2 x}{\cos^2 x \cdot 3 \cdot x^2}$$

$$\parallel$$

$$\lim_{x\to 0} \frac{\frac{\sin x - x\cos x}{\cos x}}{x^3} = \lim_{x\to 0} \frac{\sin x - x\cos x}{x^3 \cos x} = \lim_{x\to 0} \frac{1}{3\cos^2 x} \cdot \lim_{x\to 0} \frac{\sin^2 x}{x^2} = \frac{1}{3}$$

$$= \lim_{x\to 0} \frac{1}{\cos x} \cdot \lim_{x\to 0} \frac{\sin x - x\cos x}{x^3} = \lim_{x\to 0} \frac{\cancel{\cos x} - [\cancel{\cos x} + \cancel{x}(-\sin x)]}{3x^{\cancel{2}}} = \frac{1}{3}\lim \frac{\sin x}{x}$$

(5) $$\lim_{x\to \pi^-} \frac{\sin x}{1-\cos x} = \frac{0}{1-(-1)} = 0$$

$$\lim_{x\to\pi^-} \frac{(\sin x)'}{(1-\cos x)'} = \lim_{x\to\pi^-} \frac{\cos x}{\sin x} = -\infty$$