§4.7 Optimization Problems

- <u>steps in solving optimization problems</u>

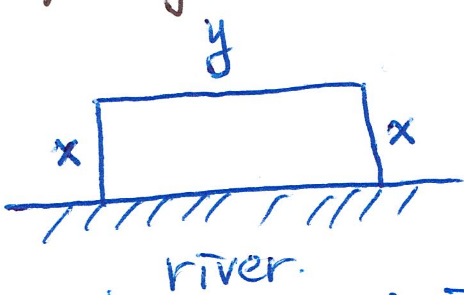
  (1) understand the problem: unknowns, given quantities, given conditions

  (2) draw a diagram: Identify the given and required quantities on the diagram

  (3) introduce notations:

  ...

<u>Ex. 1</u> A farmer has <u>2400 ft</u> of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has <u>the largest area</u>?



$2x + y = 2400$

$\Downarrow$

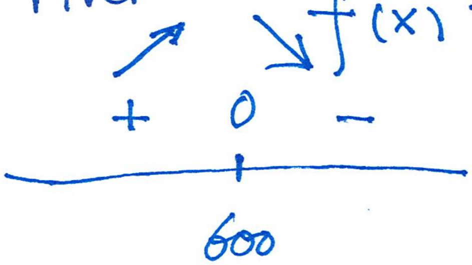
$y = 2400 - 2x$

max $A = xy$

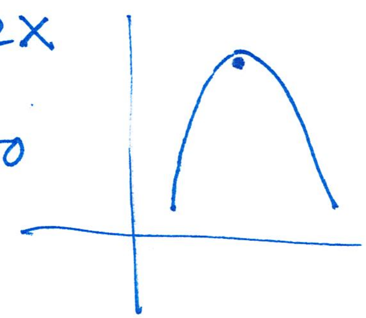
$f(x) = x(2400 - 2x)$

$f'(x) = 2400 - 4x = 0 \Longrightarrow x = \frac{2400}{4} = 600$

$= 4(600 - x)$



$$\begin{cases} x = 600 \\ y = 1200 \end{cases}$$



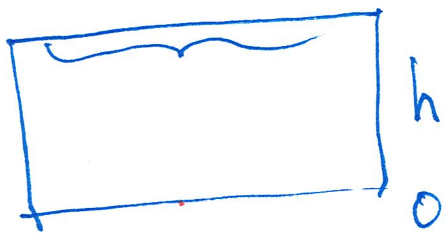
Ex. 2 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will <u>minimize</u> the cost of the metal to manufacture the can. $\boxed{1\text{ L} = 1000\text{ cm}^3}$

$$\min S = 2\pi r^2 + 2\pi r h \qquad V = \pi r^2 h = 1000$$

$$\Downarrow$$

$$h = \frac{1000}{\pi r^2}$$

$$= 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2}$$



$$f(r) = 2\pi r^2 + \frac{2000}{r}$$

$$0 = f'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2} = \frac{4[\pi r^3 - 500]}{r^2}$$

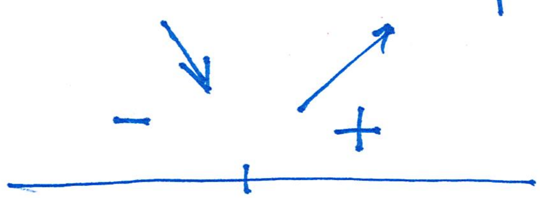
$$\Rightarrow \pi r^3 - 500 = 0 \Rightarrow 0 = r^3 - \left[\left(\frac{500}{\pi}\right)^{\frac{1}{3}}\right]^3 = \left[r - \left(\frac{500}{\pi}\right)^{\frac{1}{3}}\right]\left[r^2 + cr + c^2\right]$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

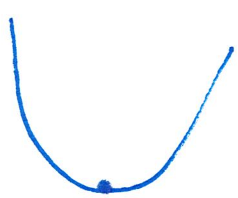
$$r = \left(\frac{500}{\pi}\right)^{\frac{1}{3}}$$

$$c^2 - 4 \cdot 1 c^2 = -3c^2$$

$$c = \left(\frac{500}{\pi}\right)^{\frac{1}{3}}$$



$$\left(\frac{500}{\pi}\right)^{\frac{1}{3}}$$



$$r = \left(\frac{500}{\pi}\right)^{\frac{1}{3}}, \quad h = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{\frac{2}{3}}}$$

1^{st} derivative test for absolute extreme values

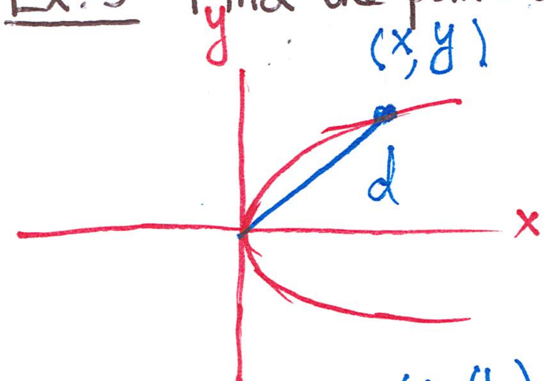
Assume that c is a critical number of a continuous function f defined on I.

(a) $f'(x)>0 \ \forall x<c$ and $f'(x)<0 \ \forall x>c \Rightarrow f(c)$ is the absolute maximum value of f

(b) $f'(x)<0 \ \forall x<c$ and $f'(x)>0 \ \forall x>c \Rightarrow f(c)$ is the absolute minimum value of f

c

Ex. 3 Find the point on the parabola $y^2=2x$ that is closest to the point $(1,4)$.



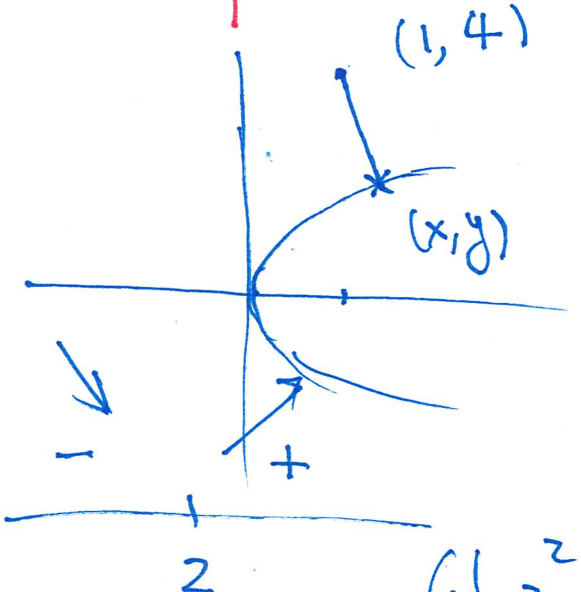
$x=\frac{1}{2}y^2$

$y^2=2x \Rightarrow x=\frac{1}{2}y^2$

$\cancel{d=\sqrt{x^2+y^2}}$

$\min d=\sqrt{(x-1)^2+(y-4)^2}=\sqrt{\left(\frac{1}{2}y^2-1\right)^2+(y-4)^2}$

$\min d^2=\min f(y)=\min \left(\frac{1}{2}y^2-1\right)^2+(y-4)^2$



$f'(y)=2\left(\frac{1}{2}y^2-1\right)\cdot y+2(y-4)$

$=y^3-8=\cancel{(y=2}\ y^3-2^3=(y-2)(y^2+2y+4)$

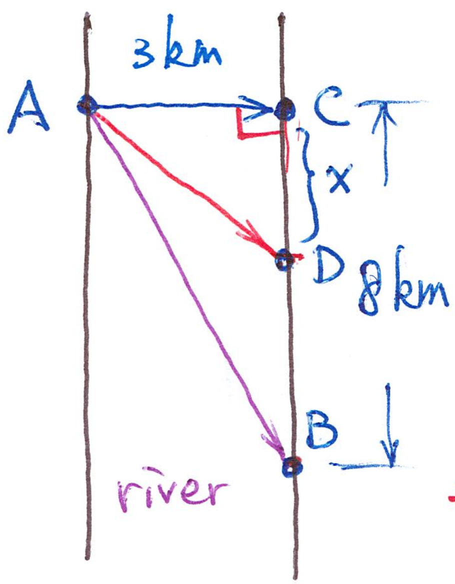
$=(y-2)\left[(y+1)^2+3\right]=0 \Rightarrow y=2$

$\left(\frac{1}{2}\cdot 2^2,\ 2\right)=(2,2)$

Ex. 4 A man lauches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possibl
He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B.
If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible
(the speed of the water is negligible.)

$d = s \cdot t$

$\Rightarrow t = \dfrac{d}{s}$



$$\min\ t = \frac{\sqrt{3^2+x^2}}{6} + \frac{8-x}{8}$$

$$f(x) = \frac{\sqrt{9+x^2}}{6} + \frac{8-x}{8}$$

$$f' = \frac{1}{6}\cdot\cancel{\frac{1}{2}}\cdot\frac{\cancel{2}x}{\sqrt{9+x^2}} - \frac{1}{8} = \frac{4x - 3\sqrt{9+x^2}}{24\sqrt{9+x^2}} = 0$$

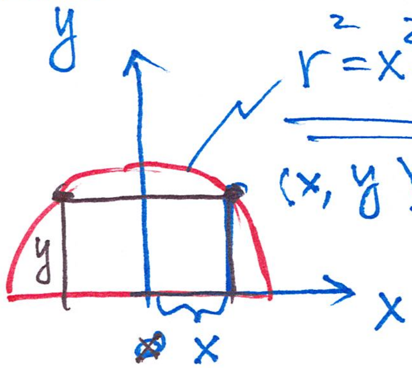
$$\Rightarrow 4x = 3\sqrt{9+x^2} \Rightarrow 16x^2 = 9(9+x^2) = 81 + 9x^2$$

$$x^2 = \frac{81}{7} \Rightarrow x = \sqrt{\frac{81}{7}} = \frac{9}{\sqrt{7}}$$

$-\quad +$

$\sqrt{\dfrac{81}{7}}$

Ex. 5 Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.



$$r^2 = x^2 + y^2 \quad \Downarrow \quad y = \sqrt{r^2 - x^2}$$

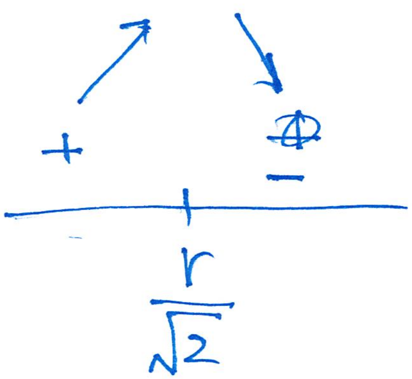
$$A = \cancel{xy}\ 2xy = 2x \cdot \sqrt{r^2 - x^2}$$

$$\Rightarrow f(x) = 2x \cdot \sqrt{r^2 - x^2}$$

$$f'(x) = 2\sqrt{r^2 - x^2} + \cancel{2}x \cdot \frac{1}{\cancel{2}\sqrt{r^2 - x^2}} \cdot (-2x)$$

$$= 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2\left[(r^2 - x^2) - x^2\right]}{\sqrt{r^2 - x^2}}$$

$$= \frac{2}{\sqrt{r^2 - x^2}} \cdot \underbrace{(r^2 - 2x^2)}_{(r + \sqrt{2}x)(r - \sqrt{2}x)} = 0 \Rightarrow x^2 = \frac{r^2}{2} \Rightarrow x = \frac{r}{\sqrt{2}}$$



$$f\left(\frac{r}{\sqrt{2}}\right) = \frac{2r}{\sqrt{2}} \cdot \sqrt{r^2 - \frac{r^2}{2}} = \frac{2r}{\sqrt{2}} \sqrt{\frac{r^2}{2}} = r^2$$