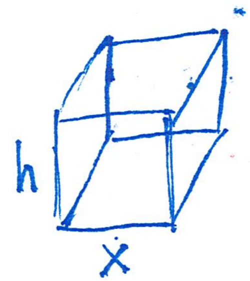
#14 (P337) A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.



Given $V = 32{,}000\ cm^3 = x^2 h \Longrightarrow h = \dfrac{32000}{x^2}$

$a^3 - b^3 = (a-b)\cdot(a^2+ab+b^2)$

$$\min S = \min\left(x^2 + 4xh\right) = \min\left(x^2 + \frac{4 \times 32000}{x}\right) = \min f(x)$$

$$f' = 2x - \frac{128{,}000}{x^2} = \frac{2x^3 - 128{,}000}{x^2} = \frac{2}{x^2}\left[x^3 - \underset{\substack{\| \\ 4^3 \cdot 10^3 = (40)^3}}{64{,}000}\right]$$

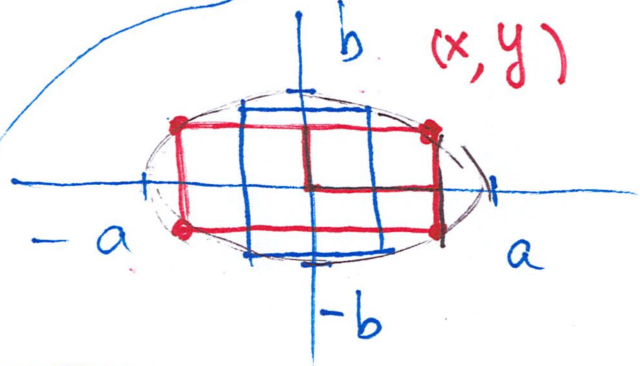
$$= \underbrace{\frac{2}{x^2}}_{>0}(x-40)\underbrace{\left(x^2 + 40x + 40^2\right)}_{>0} = 0$$

$\Longrightarrow$ $\underline{x = 40}$

f': $-$ 0 $+$ at 40, $\searrow$ $\nearrow$

$$h = \frac{32000}{(40)^2} = \frac{320}{16} = 20$$

#26 (P338) Find the area of the largest rectangle that can be inscribed in the ellipse $\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$. $\Rightarrow y=b\sqrt{1-\frac{x^2}{a^2}}$



$$A=(2x)(2y)=4xy$$

$$f(x)=4bx\sqrt{1-\frac{x^2}{a^2}}$$

max $f(x)$

$$f'(x)=4b\left[\sqrt{\frac{a^2-x^2}{a^2}}+x\,\frac{-\frac{\cancel{2}x}{a^2}}{\cancel{2}\sqrt{\frac{a^2-x^2}{a^2}}}\right]=\frac{4b\left[\frac{a^2-x^2}{a^2}-\frac{x^2}{a^2}\right]}{\sqrt{\frac{a^2-x^2}{a^2}}}$$

$$\left[\ \right]=\frac{a^2-2x^2}{a^2}, \qquad \sqrt{\frac{a^2-x^2}{a^2}}=\frac{\sqrt{a^2-x^2}}{a}$$

$$=\frac{\frac{4b}{a}(a^2-2x^2)}{\sqrt{a^2-x^2}}=0 \Rightarrow a^2-2x^2=0 \Rightarrow x=\frac{a}{\sqrt{2}}$$

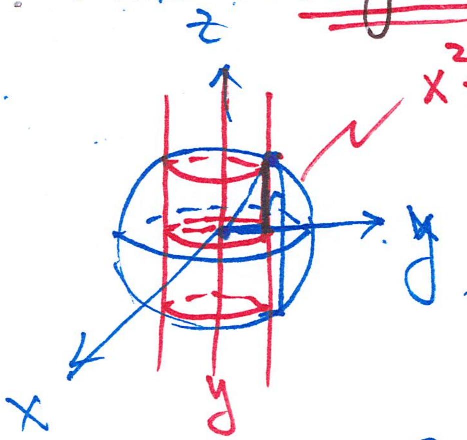
$$y=b\sqrt{1-\frac{x^2}{a^2}}=b\sqrt{1-\frac{1}{a^2}\left(\frac{a}{\sqrt{2}}\right)^2}=b/\sqrt{2}$$

$$A=4\cdot\frac{a}{\sqrt{2}}\cdot\frac{b}{\sqrt{2}}=\underline{2ab}$$

$$g(x)=\sqrt{h(x)}$$

$$g'(x)=\frac{1}{2\sqrt{h(x)}}\cdot h'(x)$$

#3 (P338) A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.



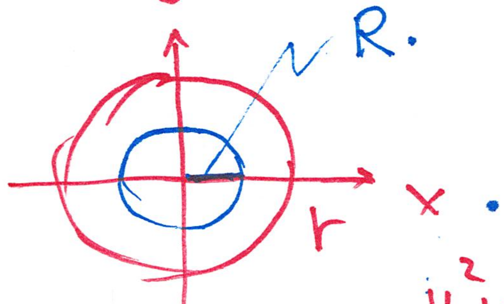
$x^2+y^2+z^2=r^2$

$y^2+z^2=r^2$

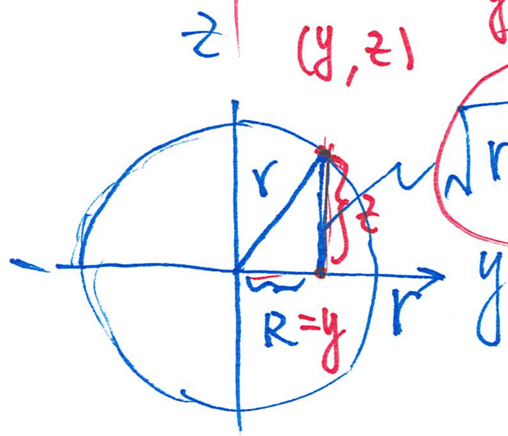
$V = \pi R^2 \cdot h = 2\pi R^2\sqrt{r^2-R^2} = f(R)$

$f'(R) = 2\pi\left[2R\sqrt{r^2-R^2} + R^2\frac{-\cancel{2}R}{\cancel{2}\sqrt{r^2-R^2}}\right]$

$= \frac{2\pi R[2(r^2-R^2)-R^2]}{\sqrt{r^2-R^2}} = \frac{2\pi R(2r^2-3R^2)}{\sqrt{r^2-R^2}}$



$y^2+z^2=r^2 \Rightarrow z=\sqrt{r^2-R^2}$ $\Rightarrow$ $R = \cancel{\frac{r}{\sqrt{2}}}\sqrt{\frac{2}{3}}\, r$

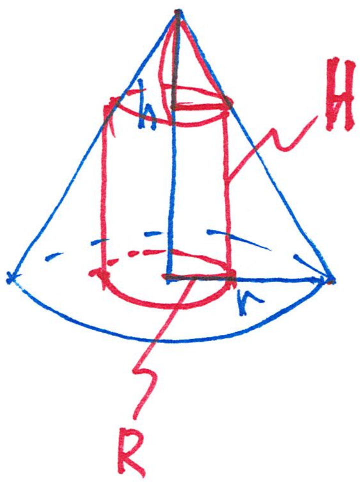


$\sqrt{r^2-R^2} = h/2$

$h = 2\sqrt{r^2-R^2}$

$\cancel{V = f\left(\frac{r}{\sqrt{2}}\right) = 2\pi\left(\frac{r}{\sqrt{2}}\right)^2\sqrt{r^2-\left(\frac{r}{\sqrt{2}}\right)^2} = \pi r^2\sqrt{\frac{r^2}{2}} = \frac{\pi r^3}{\sqrt{2}}}$

32 (P338) A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest volume of such a cylinder.



$$V = \pi R^2 H = \pi R^2 h\left(1 - \frac{R}{r}\right)$$

$$= \frac{\pi h}{r} R^2 (r - R) = f(R)$$

$$\frac{R}{r} = \frac{h-H}{h}$$

$$\Rightarrow h - H = \frac{hR}{r}$$

$$H = h - \frac{hR}{r} = h\left(1 - \frac{R}{r}\right)$$

$$f' = \frac{\pi h}{r}\left[2R(r-R) + R^2 \cdot (-1)\right]$$

$$= \frac{\pi h}{r}\left[2rR - 3R^2\right]$$

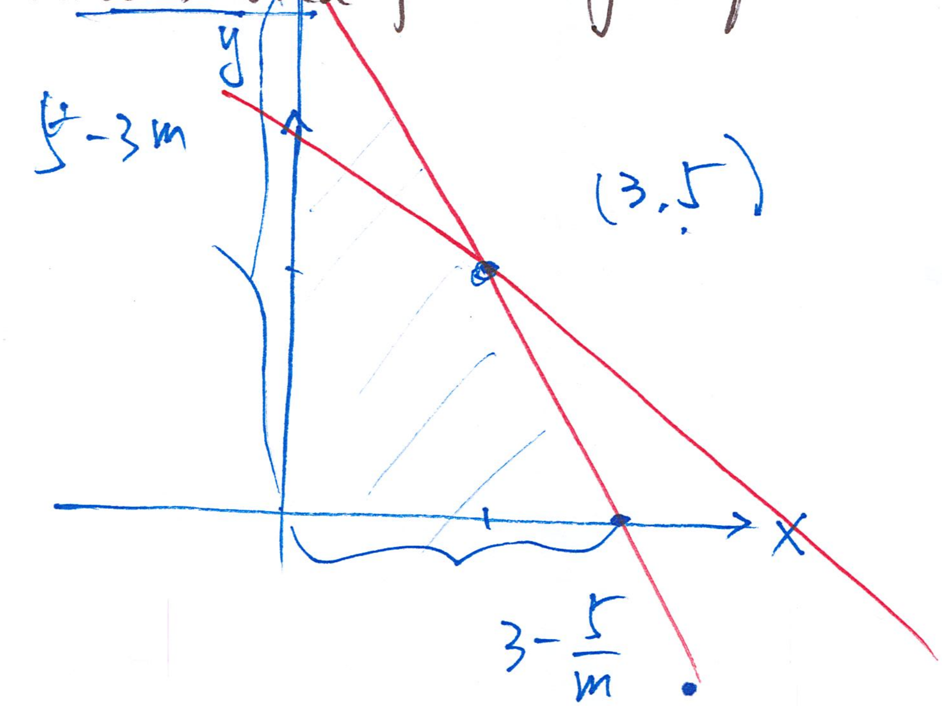
$$= \frac{\pi h}{r} R\left[2r - 3R\right] = 0$$

$$\Rightarrow R = \frac{2r}{3}$$

$$V_{max} = \frac{\pi h}{r} \cdot \left(\frac{2r}{3}\right)^2 \left(r - \frac{2}{3}r\right)$$

$$= \frac{4\pi h r^2}{27}$$

#54 (P339) Find an equation of the line through the point (3, 5) that cuts off the least area from the first quadrant.



m

$$\frac{y-5}{x-3} = m \Rightarrow y = 5 + m(x-3)$$

$$y = 0 \Rightarrow x = 3 - \frac{5}{m}$$

$$x = 0 \Rightarrow y = 5 - 3m$$

$$A = \frac{1}{2}\left(3 - \frac{5}{m}\right)(5 - 3m) = f(m)$$

$$f' = \frac{1}{2}\left[\frac{5}{m^2}(5-3m) + \left(3 - \frac{5}{m}\right)\cdot(-3)\right] = \frac{1}{2}\left[\frac{5(5-3m)}{m^2} - \frac{3(3m-5)}{m}\right]$$

$$= \frac{1}{2m^2}\left[(5-3m)\cdot(5+3m)\right] = 0 \Rightarrow m = \frac{5}{3} \text{ or } -\frac{5}{3}$$

$$\Rightarrow y = 5 - \frac{5}{3}(x-3) = 10 - \frac{5}{3}x$$