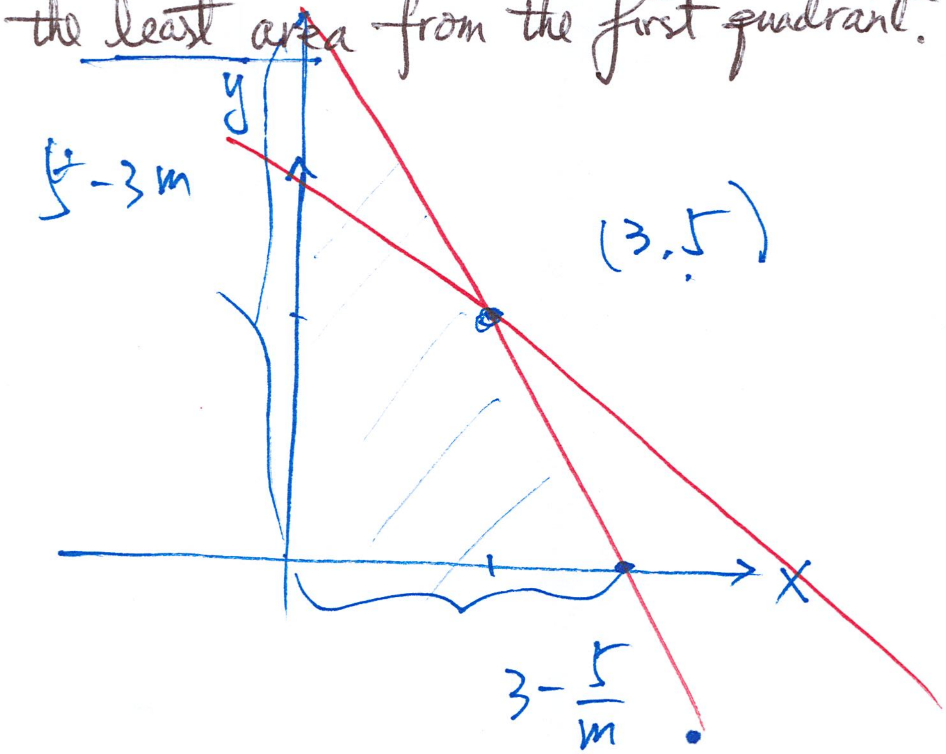
#54 (P339) Find an equation of the line through the point (3, 5) that cuts off the least area from the first quadrant.



$m = ?$ — slope

$$\frac{y-5}{x-3} = m \Rightarrow y = 5 + m(x-3)$$

$$y = 0 \Rightarrow x = 3 - \frac{5}{m}$$

$$x = 0 \Rightarrow y = 5 - 3m$$

$$A = \frac{1}{2}\left(3 - \frac{5}{m}\right)(5 - 3m) = f(m)$$

$$f' = \frac{1}{2}\left[\frac{5}{m^2}(5-3m) + \left(3 - \frac{5}{m}\right)\cdot(-3)\right] = \frac{1}{2}\left[\frac{5(5-3m)}{m^2} - \frac{3(3m-5)}{m}\right]$$

$$= \frac{1}{2m^2}\left[(5-3m)\cdot(5+3m)\right] = 0 \Rightarrow m = \frac{5}{3} \text{ or } -\frac{5}{3}$$

$$\Rightarrow y = 5 - \frac{5}{3}(x-3) = 10 - \frac{5}{3}x$$

§4.9 Antiderivatives

f'

Def. A function F is an antiderivative of f on an interval I

$$\iff \quad F'(x) = f(x) \quad \forall x \in I.$$

Thrm $F'(x) = f(x) \implies (F(x) + C)' = f(x)$ where C is a constant.

Ex. 1 Find the most general antiderivative of

(a) $f(x) = \sin x$, (b) $f(x) = \frac{1}{x}$, (c) $f(x) = x^n$, $n \neq -1$.

$F(x) = -\cos x + C$ $\qquad F(x) = \ln x + C$ $\qquad F(x) = \frac{1}{n+1} x^{n+1}$

Function $F'(x) = f(x)$ / $G'(x) = g(x)$	Particular Antider.
$c\,f(x)$	$c\,F(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^n \ (n \neq -1)$	$\frac{1}{n+1} x^{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\boxed{\frac{1}{1+x^2}}$	$\boxed{\tan^{-1} x}$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$

Ex. 2 Find all functions g such that

$$g'(x) = 4 \sin x + \frac{2x^5 - \sqrt{x}}{x}$$

$$g(x) = 4 \cdot (-\cos x) + \left[2 \cdot \frac{1}{5} x^5 - 2x^{\frac{1}{2}}\right] + C$$

Ex. 3 Find f if $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = -2$.

$$f(x) = e^x + 20 \tan^{-1} x + C$$

$$f(0) = -2 = e^0 + 20 \tan^{-1} 0 + C$$

$$C = -2 - 1 - 20 \tan^{-1} 0 = -3$$

Ex. 4 Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

$$f'(x) = 12 \cdot \frac{1}{3}x^3 + 6 \cdot \frac{1}{2}x^2 - 4x + C_1 = 4x^3 + 3x^2 - 4x + C_1$$

$$f(x) = x^4 + x^3 - 4 \cdot \frac{1}{2}x^2 + C_1 x + C_2$$

$$= x^4 + x^3 - 2x^2 + C_1 x + C_2 = x^4 + x^3 - 2x^2 - 3x + 4$$

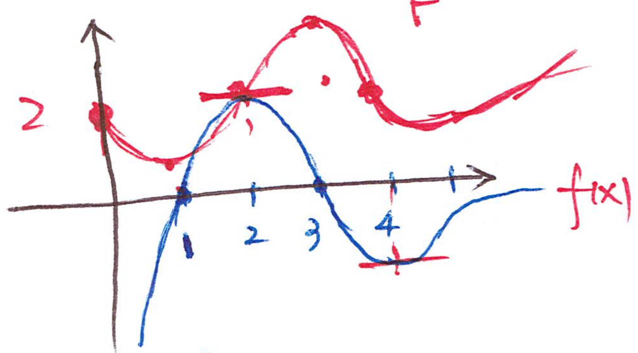
$$4 = f(0) = C_2$$

$$1 = f(1) = 1 + 1 - 2 + C_1 + 4 = C_1 + 4 \Rightarrow C_1 = -3$$

Ex. 5 The graph of f is given. Make a rough sketch of an antiderivative, given $F(0) = 2$.

F' = f

F



$F'(x) = f(x)$

	0		1		2		3		4	
		$-$	0	$+$		$+$	0	$-$		$-$

l.min   l.max

$F''(x) = f'(x)$

	0		1		2		3		4	
			$+$		0	$-$			0	$+$

- <u>rectilinear motion</u>

$$a(t) = v'(t), \quad v(t) = s'(t)$$

<u>Ex. 6</u> A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

$v(t) = 3t^2 + 4t + C, \quad -6 = v(0) = C \Rightarrow v(t) = 3t^2 + 4t - 6$

$s(t) = t^3 + 2t^2 - 6t + C, \quad 9 = s(0) = C \Rightarrow s(t) = t^3 + 2t^2 - 6t + 9$

<u>Ex. 7</u> A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

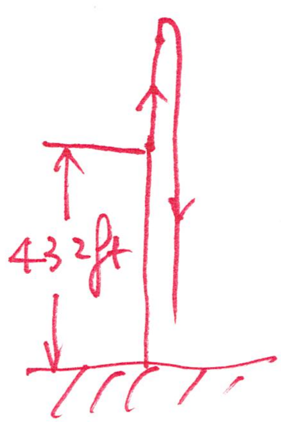
$F = mg$

$a(t) = -32 \text{ ft}/s^2, \quad v(t) = -32t + C$

$48 = v(0) = C \Rightarrow v(t) = -32t + 48$

$s(t) = -16t^2 + 48t + C, \quad 432 = s(0) = C$

$\Rightarrow s(t) = -16t^2 + 48t + 432$



$0 = v(t) = -32t + 48$

$\Rightarrow t = \frac{48}{32} = \frac{12}{8} = \frac{3}{2}$

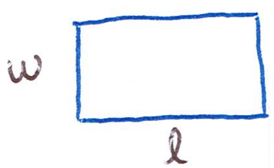
$s(\frac{3}{2}) = ?$

$s(t) = 0 \Rightarrow t = ?$

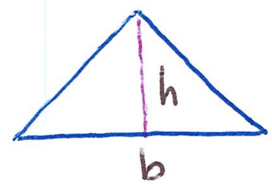
Chapter 5 Integrals (4 lectures)

§5.1 Areas and ~~Distances~~

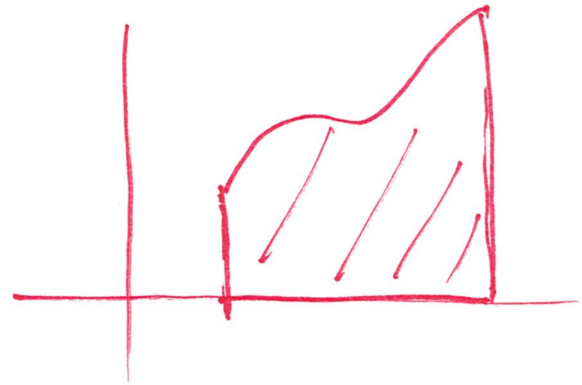
- **area problem**



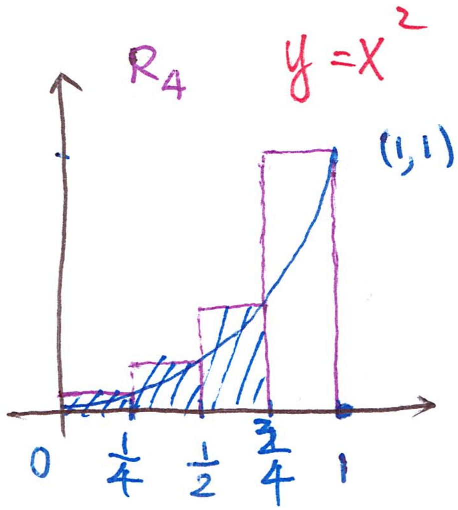
$A = wl$

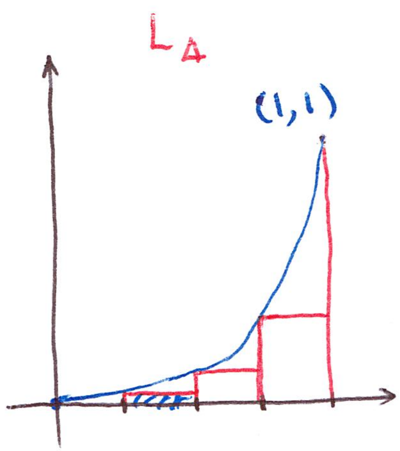


$A = \frac{1}{2} bh$



Ex. 1 Use rectangles to estimate the area under the parabola $\underset{f(x)}{\underbrace{y = x^2}}$ from 0 to 1.





$$R_4 = \frac{1}{4}\cdot f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4}\cdot f\left(\frac{3}{4}\right) + \frac{1}{4} f(1)$$

$$= \frac{1}{4}\left[\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2\right] = \frac{1^2+2^2+3^2+4}{4\cdot 16}$$

$$L_4 = \left[\frac{1}{4}\cdot 0 + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4} f\left(\frac{3}{4}\right)\right) = \frac{\frac{1}{6}4\cdot 5\cdot 9}{64}$$

$$= 0 + \frac{1}{4}\left[\left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^3\right)$$

$$= \frac{2^2 + 3^2}{4\cdot 16}$$

Ex. 2 Partition of the interval $[0,1]$:

$$\begin{array}{ccccccccc} 0 & < & \frac{1}{n} & < & \frac{2}{n} & < \cdots < & \frac{n}{n} & = 1 \\ \| & & \| & & \| & & \| & \\ x_0 & & x_1 & & x_2 & \cdots & x_n & \end{array}$$

$$\Delta x_i = x_i - x_{i-1} = \frac{1}{n}$$

Let $R_n = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n$

$L_n = f(x_0)\Delta x_1 + f(x_1)\Delta x_2 + \cdots + f(x_{n-1})\Delta x_n$

For $f(x) = x^2$, prove that $\lim\limits_{n\to\infty} R_n = \frac{1}{3}$ and that $\lim\limits_{n\to\infty} L_n = \frac{1}{3}$.

$$R_n = \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right]$$

$$= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right] = \frac{1^2 + 2^2 + \cdots + n^2}{n \cdot n^2}$$

$n \cdot n \cdot 2n$

$$= \frac{1}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1) \xrightarrow[n\to\infty]{} \frac{1}{3}$$

$$L_n = \frac{1}{n}\left[f(0) + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)\right]$$

$$= \frac{1}{n}\left[0 + \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right] = \frac{1^2 + 2^2 + \cdots + (n-1)^2}{n^3}$$

$$= \frac{1}{n^3} \cdot \frac{1}{6}(n-1)n\,(2(n-1)+1) \longrightarrow \frac{1}{3}$$

$$1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$$