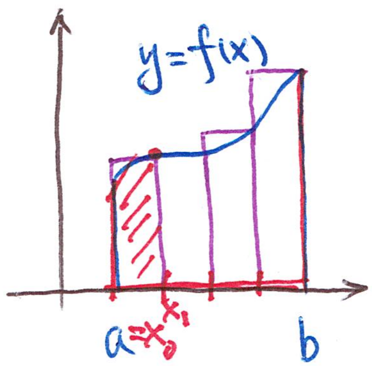


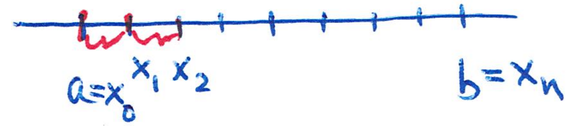
Partition of [a, b]

n - subintervals

$$a = x_0 < x_1 < \cdots < x_n = b$$

$$x_i = a + i\,\Delta x, \quad \Delta x = \frac{b-a}{n}$$

$$x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x$$



Def. The area A of the region S that lies under the graph of the continuous function f

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right]$$

$$= \lim_{n\to\infty} L_n = \lim_{n\to\infty}\left[f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x\right]$$

$$\sum_{i=0}^{n-1} f(x_i)\Delta x = f(x_0)\Delta x + \cdots + f(x_{n-1})\Delta x$$

$$= \lim_{n\to\infty}\underbrace{\left[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x\right]}_{\sum_{i=1}^{n} f(x_i^*)\Delta x}$$

$$x_i^* \in (x_{i-1}, x_i)$$

Ex. 3 Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using 4 subintervals and then 10 subintervals.

partition

$$x_i = x_0 + i\,\Delta x$$

$$= 0 + i\,\frac{2-0}{n} = \frac{2i}{n}$$

$x_2 = \frac{4}{n}$

$0 = x_0 \quad x_1 = \frac{2}{n} \quad \cdots \quad 2 = x_n$

$$A = \lim_{n \to \infty} \left(f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \right)$$

$$= \lim_{n \to \infty} \frac{2}{n} \left(e^{-\frac{2}{n}} + e^{-\frac{4}{n}} + \cdots + e^{-\frac{2n}{n}} \right)$$

§5.2 The Definite Integral

Definition of a Definite Integral f is a function defined on $[a, b]$,

partition

let $a = x_0 < x_1 < \cdots < x_n = b$ with $x_i = a + i\Delta x$, $\Delta x = \frac{b-a}{n}$ be a partition of $[a, b]$,

let $x_i^* \in [x_{i-1}, x_i]$ be sample points,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that the limit exists independent of choices of x_i^*.

f is integrable on $[a, b]$ $\Longleftrightarrow$ the limit exists.

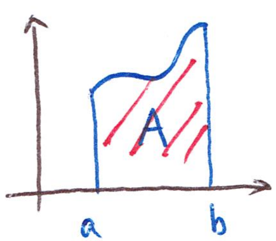
a — lower limit

b — upper limit

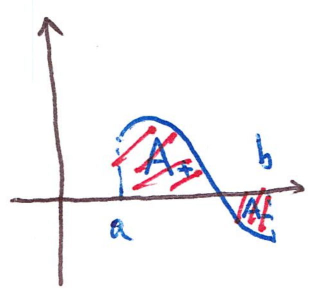
f — integrand

$\int$ — integral sign

$\sum_{i=1}^{n} f(x_i^*)\Delta x$ — Riemann sum



$A = \int_a^b f(x)dx$



$\int_a^b f(x)dx = A_+ - A_-$

Theorem f is continuous on $[a,b]$ or f has only a finite number of jump discontinuities

$\lim_{x\to c} f(x) = f(c)$

$\Longrightarrow$ f is integrable on $[a,b]$. $\Longleftarrow$ f is cont. $\Longleftarrow$ f is diff

Theorem f is integrable on $[a,b]$

$$\Longrightarrow \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \qquad \Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x.$$

Ex. 1 Express $\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^3 + x_i \sin x_i\right) \Delta x$ as an integral on the interval $[0,\pi]$.

$$= \int_0^{\pi} (x^3 + x\sin x)\,dx$$

- evaluating integrals

identities

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1), \quad \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1), \quad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Ex. 2 (a) Evaluate the Riemann sum for $f(x)=x^3-6x$, taking the sample points to be right endpoints and $a=0$, $b=3$, and $n=6$.

(b) Evaluate $\int_0^3 (x^3-6x)\,dx$.

$$f(x)=x^3-6x$$

$$\Delta x = \frac{3-0}{n} = \frac{3}{n}$$

$$x_i = a + i\Delta x$$

(b) $$\int_0^3 (x^3-6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\cdot\frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{3i}{n}\right)^3 - 6\cdot\frac{3i}{n}\right)\cdot\frac{3}{n}$$

$$= 0 + \frac{3i}{n} = \frac{3i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{3^3 i^3}{n^3} - 6\cdot\frac{3}{n} i\right]\cdot\frac{3}{n} = \lim_{n\to\infty} \frac{3}{n}\left[\left(\frac{3}{n}\right)^3 \sum_{i=1}^{n} i^3 - \frac{18}{n}\sum_{i=1}^{n} i\right]$$

$$\sum (a_i+b_i) = \sum a_i + \sum b_i$$

$$\sum_{i=1}^{n} i = 1+2+\cdots+n$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\left(\frac{3}{n}\right)^3\cdot\left(\frac{n(n+1)}{2}\right)^2 - \frac{18}{n}\,\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[\left(\frac{3}{n}\right)^4\cdot\left(\frac{n(n+1)}{2}\right)^2 - \frac{3\cdot 18\, n(n+1)}{2n^2}\right]$$

$$= \frac{3^4}{2^2} - \frac{3\cdot 18}{2}$$

Ex. 3 (a) Set up an expression for $\int_1^3 e^x\,dx$ as a limit of sums. $\Delta x = \frac{3-1}{n} = \frac{2}{n}$

$x_i = 1 + i\frac{2}{n} = \frac{n+2i}{n}$

(b) Use $\sum_{i=1}^{n} e^{1+\frac{2i}{n}} \overset{?}{=} \left[e^{\frac{3n+2}{n}} - e^{\frac{n+2}{n}}\right] / (e^{\frac{2}{n}} - 1)$ to evaluate the expression.

$$\int_1^3 e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{\infty} e^{\frac{n+2i}{n}} \cdot \frac{2}{n} = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{\infty} e^{1+\frac{2i}{n}} =$$

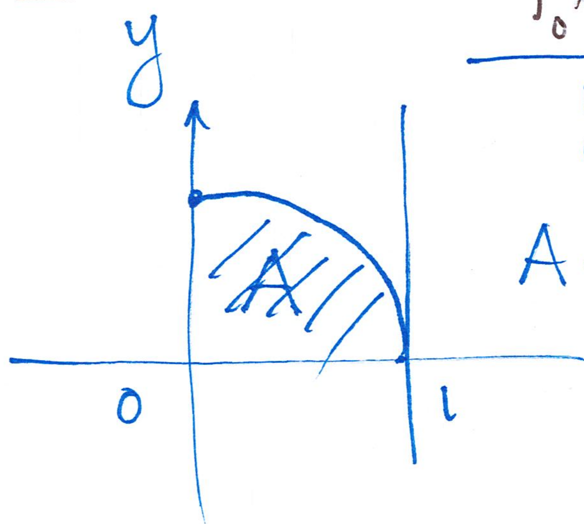
$$= \lim_{n\to\infty} \frac{2}{\underset{\to\infty}{n}} \cdot \frac{e^{\frac{3n+2}{n}} - e^{\frac{n+2}{n}}}{\underset{\to 0}{e^{\frac{2}{n}} - 1}} \overset{?}{=} 2(e^3 - e) \lim_{n\to\infty} \frac{1}{n(e^{\frac{2}{n}} - 1)} \to \frac{1}{2}$$

$$\infty \cdot 0$$

$$\lim_{n\to\infty} \frac{e^{\frac{2}{n}} - 1}{\frac{1}{n}} = \lim \frac{-\frac{2}{n^2} \cdot e^{2/n}}{-\frac{1}{n^2}}$$

$$= e^3 - e$$

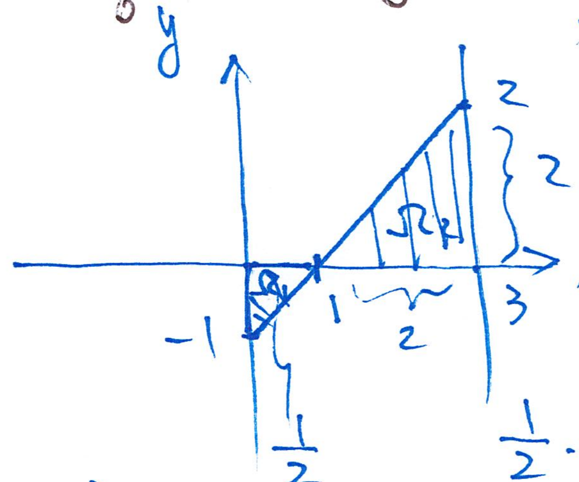
Ex. 4 Evaluate (a) $\int_0^1 \sqrt{1-x^2}\,dx$ and (b) $\int_0^3 (x-1)\,dx$ by interpreting each in terms of areas.



$y = \sqrt{1-x^2}$

$A = \frac{1}{4} \cdot \pi \cdot 1^2$

$x = \frac{\pi}{4}$



$y = x - 1$

$y = 0 = x - 1$

$\Rightarrow x = 1$

$\frac{1}{2} \cdot 2 \cdot 2 = 2$

$$\int_0^3 (x-1)\,dx = 2 - \frac{1}{2} = \frac{3}{2}$$

§5.3 The Fundamental Theorem of Calculus

a connection between differential calculus and integral calculus

The Fundamental Theorem of Calculus, Part I

Assume that f is continuous on $[a,b]$ $\Longrightarrow$ $\boxed{g(x)=\int_a^x f(t)\,dt}$ is continuous on $[a,b]$ and differentiable on (a,b).

Moreover, $g'(x)=f(x)=\frac{d}{dx}\int_a^x f(t)\,dt$

$$\| $$

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)\,dt-\int_a^x f(t)\,dt\right]$$

$$\int_x^{x+h} f(t)\,dt$$

Proof

- $$\frac{g(x+h)-g(x)}{h}=\frac{1}{h}\int_x^{x+h} f(t)\,dt$$

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt \overset{?}{=} f(x)$$

- On $[x, x+h]$ for $h>0$, $\exists\, u, v \in [x, x+h]$ such that

$$f(u) \leq \frac{g(x+h)-g(x)}{h} \leq f(v)$$

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(x)\,dx$$

<u>Ex. 2</u> Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2}\,dt$.

$$\frac{d\,g(x)}{dx} = \sqrt{1+x^2}$$

Ex. 3 Fresnel function

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

$$g(x) = \int_1^x \sec t \, dt$$

Ex. 4 Find $\frac{d}{dx}\boxed{\int_1^{x^4} \sec t \, dt.} = \sec x^4 \cdot (x^4)' = 4x^3 \sec x^4$

$$\| $$

$$g(x^4)$$

$$\frac{d}{dx} g(x^4) = g'(x^4) \cdot (x^4)'$$

The Fundamental Theorem of Calculus, Part II

Assume that f is continuous on $[a,b]$ and that $F'(x) = f(x)$ $\Longrightarrow$ $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$ $= F(x)\Big|_a^b$

Proof Let $g(x) = \int_a^x f(t)\,dt$ $\quad g'(x) = f(x)$

- $F(x) = g(x) + C$ on $[a,b]$ $\quad g(x)$ is an antider. of $f(x)$

- $F(b) - F(a) = g(b) + C - (g(a) + C) = g(b) - g(a)$

$$= \int_a^b f(x)\,dx - \cancel{\int_a^a f(x)\,dx}^{\,0}$$

Ex. 5 Evaluate the integral $\int_1^3 e^x\,dx = e^x\Big|_1^3 = e^3 - e$

$$e^x = \left(e^x\right)'$$