Fundamental Thrm of Calculus

Assume that $f(x)$ is continuous on $[a, b]$

$\Longrightarrow$ (1) $g(x) = \int_a^x f(t)\,dt$ is cont. and diff. on $[a, b]$

moreover, $\boxed{g'(x) = f(x)}$

(2) $\int_a^b f(x)\,dx \xlongequal{F'(x) = f(x)} F(b) - F(a)$

$$g(x)=\int_a^x f(t)\,dt$$

$$g(x)=\int_a^{x^4} f(t)\,dt = g(x^4) \Rightarrow \frac{d}{dx}g(x^4)=g'(x^4)\cdot(x^4)'$$

$$\Longrightarrow g'(x)=f(x^4)\cdot 4x^3$$

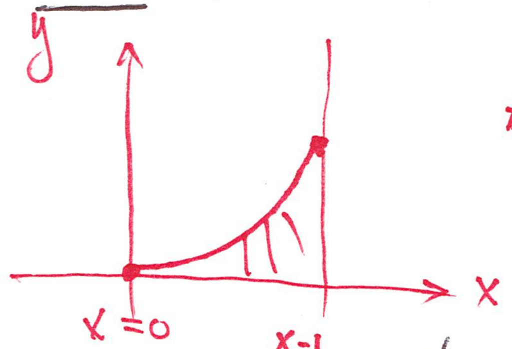
$$g(x)=\int_{h(x)}^{k(x)} f(t)\,dt = \int_{h(x)}^{a} f(t)\,dt + \int_a^{k(x)} f(t)\,dt$$

$$= -\int_a^{h(x)} f(t) + \int_a^{k(x)} f(t)\,dt$$

$$g'(x) = -f(h(x))\cdot h'(x) + f(k(x))\cdot k'(x)$$

$$= f(k(x))\cdot k'(x) - f(h(x))\cdot h'(x)$$

Ex. 6 Find the area under the parabola $y=x^2$ from 0 to 1.

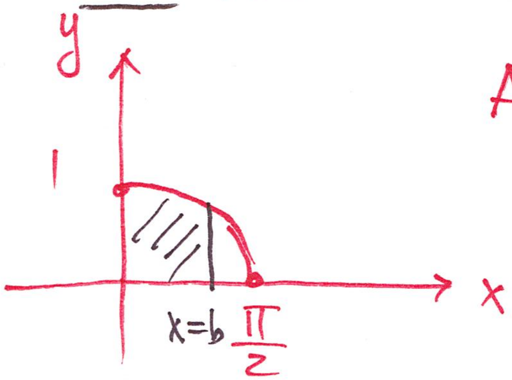


$$A=\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}\left[1^3-0^3\right]=\frac{1}{3}$$

Ex. 7 Evaluate $\int_3^6 \frac{1}{x}\,dx$. $= \ln x\Big|_3^6 = \underline{\ln 6} - \ln 3 \ne \boxed{\cancel{\ln 2 \ne \ln 3 -}}$

$$= \ln\frac{6}{3} = \ln 2$$

Ex. 8 Find the area under the cosine curve from 0 to b, where $0 \le b \le \frac{\pi}{2}$.



$$A=\int_0^b \cos x\,dx = \sin x\Big|_0^b = \sin b - \sin 0 = \sin b$$

$$\int_{-1}^0 \frac{1}{x^2}\,dx + \int_0^3 \frac{1}{x^2}\,dx \qquad \text{improper}$$

Ex. 9 What is wrong with the following calculation?

$$\int_{-1}^3 \frac{1}{x^2}\,dx = \left[-x^{-1}\right]_{-1}^3 = -\left[\frac{1}{3}+1\right] = -\frac{4}{3}.$$

- differentiation and integration as inverse processes

Fundamental Thrm of Calculus Assume that f is continuous on $[a, b]$

$\Longrightarrow$ (1) $\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$

(2) $\int_a^b F'(x)\,dx = F(b) - F(a)$

$$\int F'(x)\,dx = F(x) + C$$

§5.4 Indefinite Integrals and the Net Change Theorem

- indefinite integral

(F(x) is an integral of f(x)) $\int f(x)\,dx = F(x) \iff F'(x) = f(x)$ (F(x) is an antider. of f(x

table of indefinite integrals

$$\left[\begin{array}{l}\int c\,f(x)\,dx = c\int f(x)\,dx \\ \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx\end{array}\right.$$

$$\int k\,\cancel{x}\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (a^x)' = a^x \cdot \ln a$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \frac{1}{x^2}\,dx = \begin{cases} -\frac{1}{x} + C_1 & \text{if } x<0 \\ -\frac{1}{x} + C_2 & \text{if } x>0. \end{cases}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

Ex. 1 $\int (10x^4 - 2\sec^2 x)\,dx = 10\int \cancel{10}\, x^4\,dx - 2\int \sec^2 x\,dx$

$= 10 \cdot x^5 \cdot \frac{1}{5} - 2\tan x + C$

Ex. 2 $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\,d\theta = \int \csc\theta \cdot \cot\theta\,d\theta = -\csc x + C$

$= \int \frac{d\sin\theta}{\sin^2\theta}$ $\quad d\sin\theta = \cos\theta\,d\theta$ $\quad = -\frac{1}{\sin\theta} + C$

Ex. 3 $\int_0^3 (x^3 - 6x)\,dx = \int_0^3 x^3\,dx - 6\int_0^3 x\,dx = \left[\frac{1}{4}x^4 - 6\cdot\frac{1}{2}x^2\right]_0^3$

$= \frac{1}{4}\left[3^4 - 0^4\right] - 3\left[3^2 - 0^2\right] = \frac{1}{4}\cdot 3^4 - 3^3 = -\frac{1}{4}\cdot 3^3.$

$= \left(\frac{3}{4} - 1\right)\cdot 3^3$

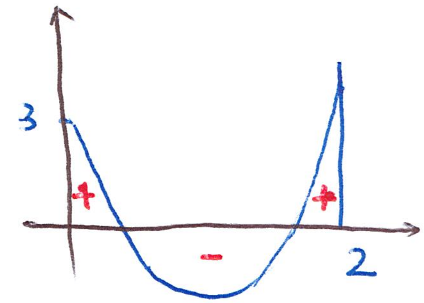
Ex. 4 $\displaystyle\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$ and interpret the result in terms of areas.

$$= 2\int_0^2 x^3\,dx - 6\int_0^2 x\,dx + 3\int_0^2 \frac{1}{x^2+1}\,dx$$

$$= \left[\frac{1}{2}x^4 - 3x^2 + 3\arctan x\right]_0^2$$

$$= \left(8 - 12 + 3\arctan 2\right)$$

$$= -4 + 3\arctan 2$$



Ex. 5 $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int_1^9 \left(2 + \sqrt{t} - \frac{1}{t^2}\right) dt$

$$= \left[2t + \frac{2}{3}t^{\frac{3}{2}} + \frac{1}{t}\right]_1^9$$

- applications

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

(see examples

$$d = \int_1^4 \boxed{|v(t)|\,dt} = \int_1^3 (-t^2+t+6)\,dt + \int_3^4 (t^2-t-6)\,dt$$

$$s'(t) = v(t)$$

Ex. 6 A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ m/s $= (t-3)(t+2)$

(a) Find the displacement of the particle during the time period $1 \le t \le 4$

(b) Find the distance traveled during this time period.

(a) $s(t) = \int v(t)\,dt = \int (t^2 - t - 6)\,dt = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t + C$

(b) $d\ \left|s(3) - s(1)\right| = \left|\left(\frac{1}{3}\cdot 3^3 - \frac{1}{2}\cdot 3^2 - 6\cdot 3\right) - \left(\frac{1}{3} - \frac{1}{2} - 6\right)\right|$

$- \quad 0 \quad +$

$1 \quad 3 \quad 4$

$\left|s(4) - s(3)\right| = \left|\left(\frac{1}{3}\cdot 4^3 - \frac{1}{2}\cdot 4^2 - 6\cdot 4\right) - (\quad)\right|$

§5.5 The Substitution Rule

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,dx^2 = \int \sqrt{1+x^2}\,d\left(1+x^2\right)$$

$$2x\,dx = d(x^2) = (x^2)'dx$$

$$\overset{u=1+x^2}{=\!=} \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + C$$

The Substitution Rule

$u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I $\Longrightarrow$ $\int f(g(x))\underbrace{g'(x)\,dx}_{dg(x)} = \int f(u)\,du$

Ex. 1 $\int x^3\cos(x^4+2)\,dx \;\underset{du=4x^3dx}{\overset{u=x^4+2}{=\!=}}\; \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C$

$x^3dx = \frac{1}{4}du$

$$= \frac{1}{4}\sin\left(x^4+2\right) + C$$