§1.4 Exponential Functions

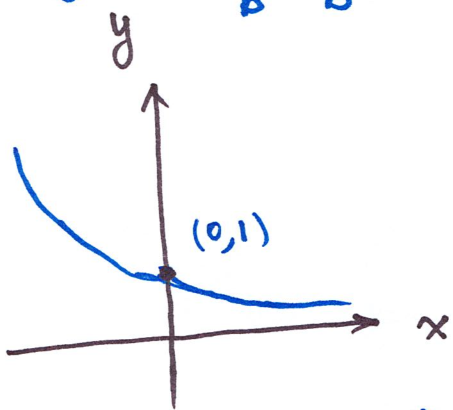
$b^3 = b \cdot \underbrace{b \cdot b} = b \cdot b^2$

$b^{-2} = b^{-1} \cdot b^{-1} = \frac{1}{b} \cdot \frac{1}{b}$

$f(x) = b^x$ ~ exponent

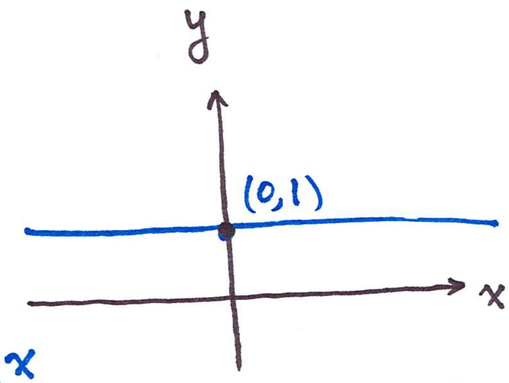
base $b > 0$

$\mathrm{dom}(f) = (-\infty, +\infty)$



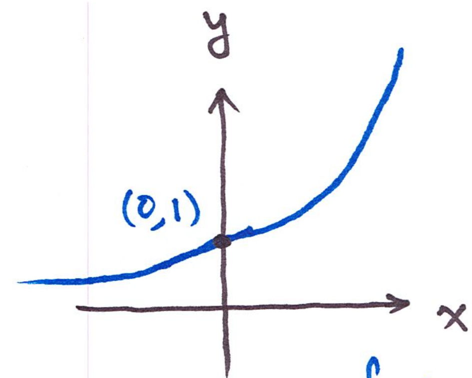
$0 < b < 1$ $\quad f(x) = \left(\frac{1}{2}\right)^x$

$\mathrm{range}(f) = (0, +\infty)$



$b = 1$ $\quad f(x) = 1^x = 1$

$\mathrm{range}(f) = \{1\}$



$b > 1$ $\quad f(x) = 2^x$

$\mathrm{range}(f) = (0, +\infty)$

Law of Exponents

$$b^{x+y} = b^x \cdot b^y$$

$$b^{x-y} = b^x \cdot b^{-y} = \frac{b^x}{b^y}$$

$$\left(b^x\right)^y = b^{x \cdot y}$$

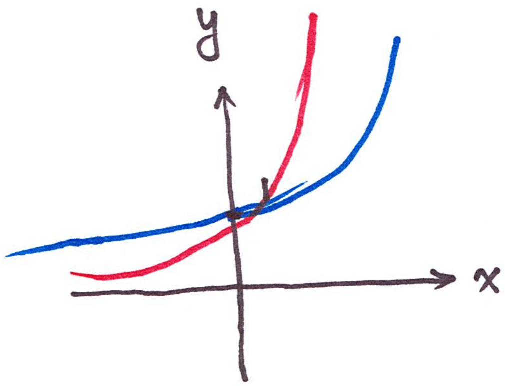
$$(ab)^x = a^x \cdot b^x$$

The Number e

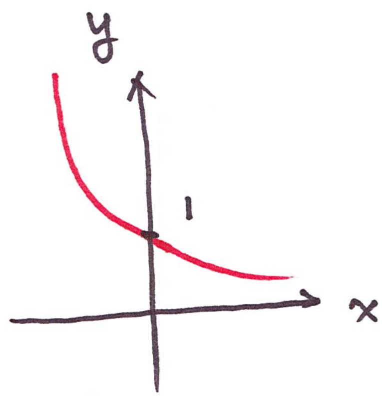
$$e \approx 2.71828$$

$$\left.\frac{d}{dx}2^x\right|_{x=0} \approx 0.7, \quad \left.\frac{d}{dx}3^x\right|_{x=0} \approx 1.1, \quad \left.\frac{d}{dx}e^x\right|_{x=0} = 1$$

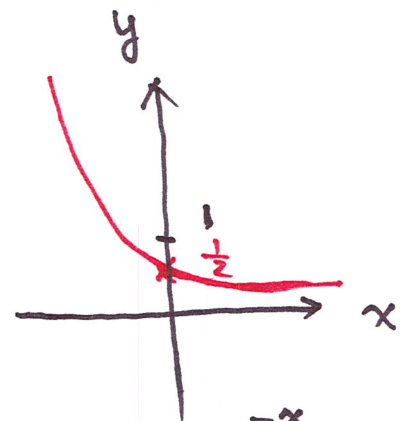
Ex. 4 Sketch $y = \frac{1}{2}e^{-x} - 1$, dom $= ?$, ~~dom~~ range $= ?$



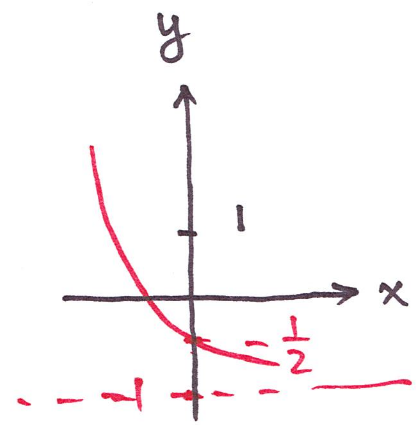
$y = e^x$



$y = e^{-x}$



$y = \frac{1}{2}e^{-x}$



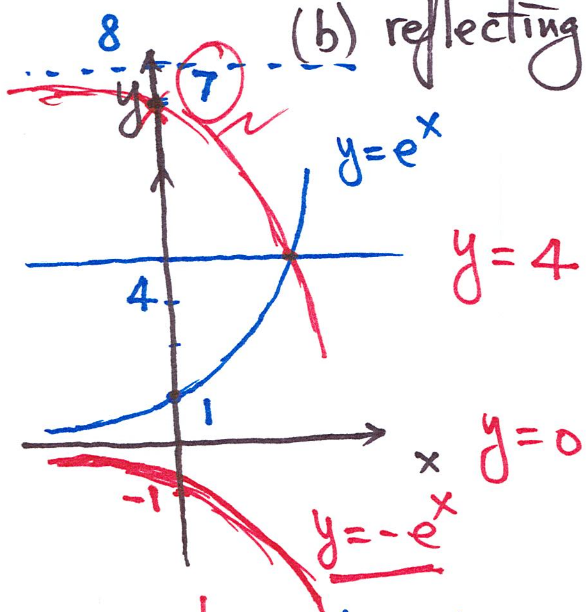
$y = \frac{1}{2}e^{-x} - 1$

$$\mathrm{dom}(f) = (-\infty, +\infty), \quad \mathrm{range}(f) = (-1, +\infty)$$

#18 (P53) Starting with the graph of $y=e^x$, find the equation of the graph that results from

(a) reflecting about the line $y=4$

(b) reflecting about the line $x=2$

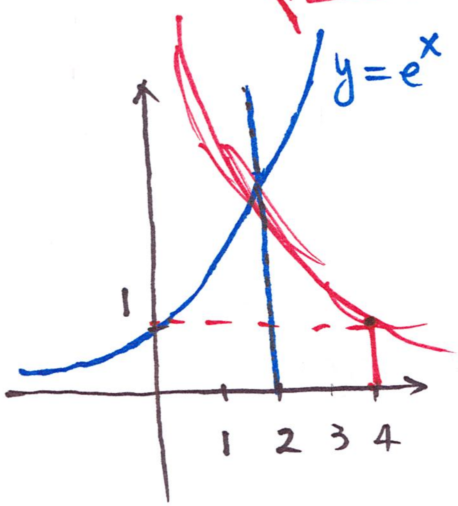


$$y \neq 4 - e^x \qquad y(0) = 4 - e^0 = 3$$

$$y = 8 - e^x \qquad y(0) = 8 - e^0 = 7$$

$y=f(x)$ reflecting w.r.t. $y=b$

$$\Rightarrow \boxed{y = 2b - f(x)}$$

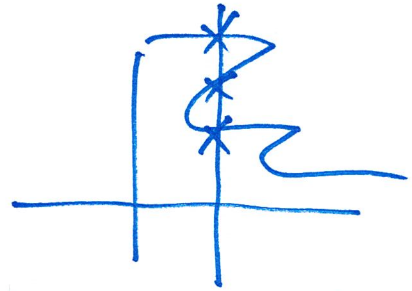


$$y = e^{-x} - 4$$

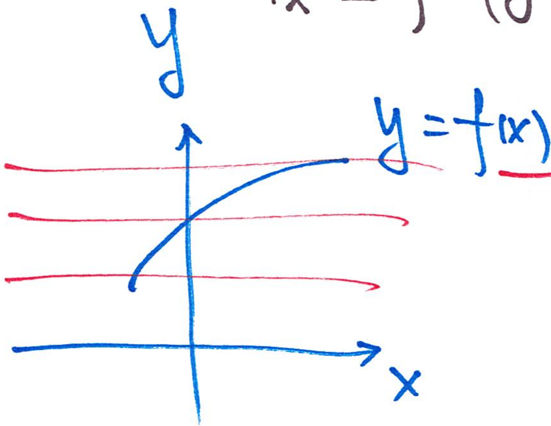
$$y = e^{-x+4}$$

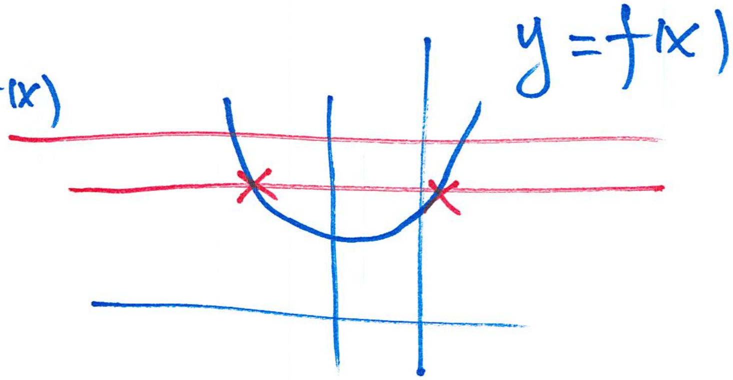
§1.5 Inverse Functions and Logarithms

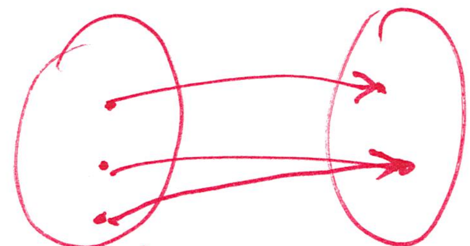
$\underline{x=f^{-1}(y)}$ $\quad y=f(x)$ $\quad$ single valued



$x=f^{-1}(y)$ $\quad$ what guarantees f^{-1} being single valued?







not 1-to-1

Def (1-to-1) $\quad$ $f(x)$ is a 1-to-1 function

$$\Longleftrightarrow \forall\, x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$$

$$\Longleftrightarrow f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$$

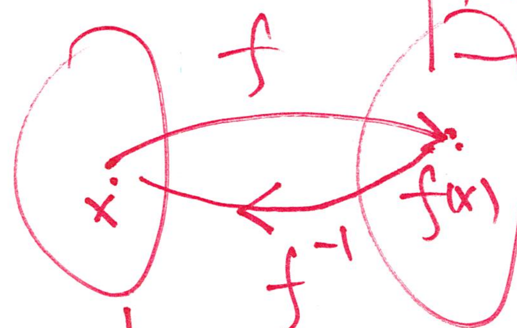
$\Longleftrightarrow$ horizontal line test.

Def. (inverse function) Assume that f is a 1-to-1 function with $\mathrm{dom}(f)=A$, $\mathrm{range}(f)=B$, the inverse function f^{-1} is defined by

$$f^{-1}(y)=x \quad \forall y\in B \iff f(x)=y$$

$$y=f\left(f^{-1}(y)\right)=f(x)$$

$$\boxed{x=f^{-1}\left(f(x)\right)=f^{-1}(y)}$$



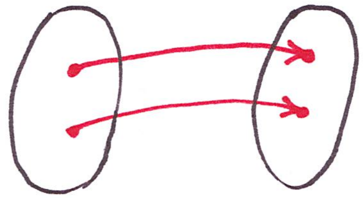
(1) $y=x^3+2 \Longrightarrow x^3=y-2$

$$x=(y-2)^{\frac{1}{3}}=f^{-1}(y)$$

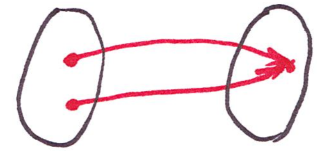
Ex.4 Find the inverse function of (1) $f(x)=x^3+2$, (2) $f(x)=x^2-x$ for $\boxed{x\geq\frac{1}{2}}$

$$y=x^2-x=\left(x-\frac{1}{2}\right)^2-\frac{1}{4}$$

$$\left(x-\frac{1}{2}\right)^2=y+\frac{1}{4} \Longrightarrow x=\frac{1}{2}+\sqrt{y+\frac{1}{4}}$$



1-to-1

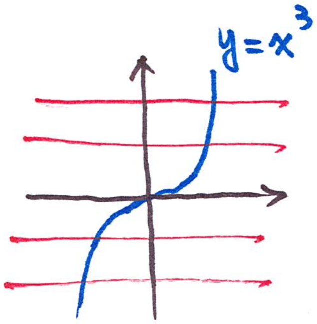


not 1-to-1

$\forall x_1, x_2$

$f(x_1) = f(x_2)$

Ex. 1 Is $f(x) = x^3$ 1-to-1?

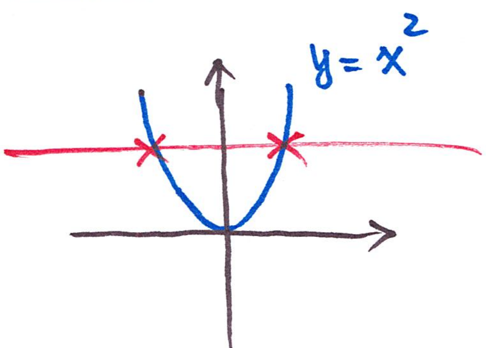


$$x_1^3 = x_2^3 \overset{?}{\Longrightarrow} x_1 = x_2$$

$$0 = x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2)$$

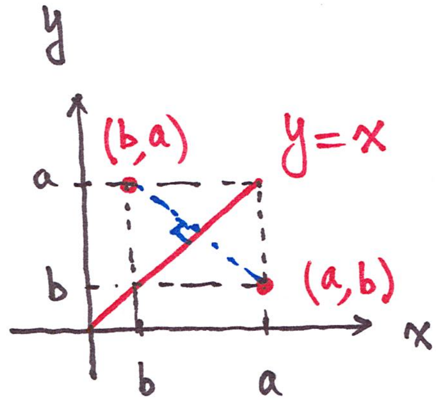
$$\Rightarrow \underline{\underline{x_1 - x_2 = 0}} \quad \text{or} \quad x_1^2 + x_1 x_2 + x_2^2 = 0$$

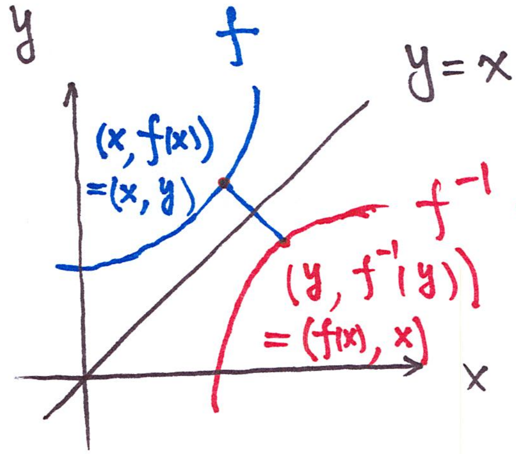
Ex. 2 Is $g(x) = x^2$ 1-to-1?



$$\cancel{f(1)} \quad 1^2 = (-1)^2 = 1$$

graph f^{-1} from graph f





f^{-1} — reflecting of f w.r.t. the line $y=x$

Logarithmic Functions

$\underline{y} = b^x > 0 \qquad x = f^{-1}(y) = \log_b \underline{y} \qquad y > 0$

$$\underline{y = \log_b x} \quad \text{for } \underline{x > 0}$$

Law of Logarithms for $x, y > 0$

$$\log_b (xy) \stackrel{?}{=} \log_b x + \log_b y$$

$$xy = b^{\log_b (xy)} \qquad b^{\log_b x + \log_b y} = b^{\log_b x} \cdot b^{\log_b y} = x \cdot y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b x^r = r \log_b x$$

Natural Logarithms $\ln x = \log_e x$

- $\log_b x = \dfrac{\ln x}{\ln b}$

- $\ln e = 1$