Ex. 2 $\int \sqrt{2x+1}\,dx \xrightarrow[du=2dx]{u=2x+1} \int u^{\frac{1}{2}} \frac{1}{2}\,du$

$= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$

$\int f(g(x))\,\underline{g'(x)\,dx} \xrightarrow[du=g'(x)dx]{u=g(x)} \int f(u)\,du$

Ex. 3 $\int \frac{x}{\sqrt{1-4x^2}}\,dx \xrightarrow[du=-8x\,dx]{u=1-4x^2} \int u^{-\frac{1}{2}} \boxed{\begin{array}{c}\left(-\frac{1}{8}\right)du \\ \parallel \\ x\,dx\end{array}}$

$= -\frac{1}{8}\cdot 2\cdot u^{\frac{1}{2}} = -\frac{1}{4}(1-4x^2)^{\frac{1}{2}} + C$

Ex. 4 $\int \underline{e}^{5x}\,dx \xrightarrow[du=5dx]{u=5x} \frac{1}{5}\int \underline{e}^{u}\,du = \frac{1}{5}e^{u} = \frac{1}{5}e^{5x} + C$

$\Downarrow$

$dx = \frac{1}{5}du$

Ex. 5 $\int \sqrt{1+x^2}\, x^5\, dx$

$x^5 dx = x^4 \cdot x\,dx$

$$\frac{u = 1+x^2}{du = 2x\,dx}$$

$\underbrace{du = 2x\,dx}$

$\Downarrow$

$x\,dx = \frac{1}{2} du$

$u = 1+x^2 \Longrightarrow x^2 = u-1$

$x^4 = (x^2)^2 = (u-1)^2$

$$\int u^{\frac{1}{2}} (u-1)^2 \frac{1}{2} du = \frac{1}{2}\int u^{\frac{1}{2}}\left[u^2 - 2u + 1\right] du$$

$$= \frac{1}{2}\int \left[u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right]$$

$$= \frac{1}{2}\left[\frac{2}{7}u^{\frac{7}{2}} - 2\cdot\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]$$

$$= \frac{1}{7}u^{\frac{7}{2}} - \frac{2}{5}u^{\frac{5}{2}} + \frac{1}{3}u^{\frac{3}{2}} + C$$

Ex. 6 $\int \tan x\, dx$

$$= \int \frac{\sin x}{\cos x} dx \quad \frac{u = \cos x}{du = -\sin x\, dx} \quad \int \frac{1}{u}(-du)$$

$$= -\int u^{-1} du = -\ln|u| = -\ln|\cos x| + C$$

- <u>definite integrals</u>

$$\int_a^b f(g(x))\,g'(x)\,dx \;\underset{du=g'(x)dx}{\overset{u=g(x):\ g(a)\to g(b)}{=\!=}}\; \int_{g(a)}^{g(b)} f(u)\,du$$

<u>Proof</u> Let $F'=f$

$$\Rightarrow \left(F(g(x))\right)' = F'(g(x))\,g'(x)$$

$\Rightarrow$ LHS =

RHS =

<u>Ex. 7</u> $\int_0^4 \sqrt{2x+1}\,\underline{dx} \;\underset{\underline{du=2dx}}{\overset{u=2x+1\,:\,1\to 9}{=\!=}}\; \int_1^9 u^{\frac{1}{2}}\,\frac{1}{2}\,du$

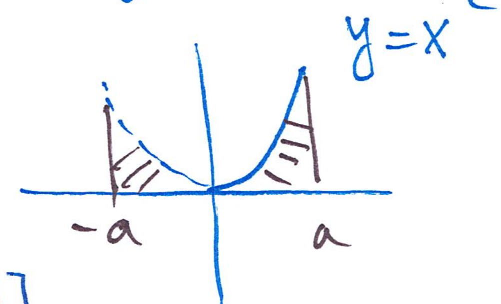
$$= \frac{1}{2}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}}\Big|_1^9 = \frac{1}{3}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \frac{1}{3}\left[3^3 - 1\right]$$

$$= \frac{26}{3}$$

Ex. 8 $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2} \;\underset{du=-5dx}{\overset{u=3-5x \,:\, -2 \to -7}{=\!=\!=}}\; \int_{-2}^{-7} u^{-2}\left(-\frac{1}{5}\right)du$

$\displaystyle = -\frac{1}{5}\cdot(-1)\, u^{-1}\Big|_{-2}^{-7} = \frac{1}{5}\left[\frac{1}{-7} - \frac{1}{-2}\right] = \frac{1}{5}\left[\frac{1}{2} - \frac{1}{7}\right]$

Ex. 9 $\displaystyle\int_1^e \frac{\ln x}{x}\,dx \;\underset{du=\frac{1}{x}dx}{\overset{u=\ln x \,:\, 0 \to 1}{=\!=\!=}}\; \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}$

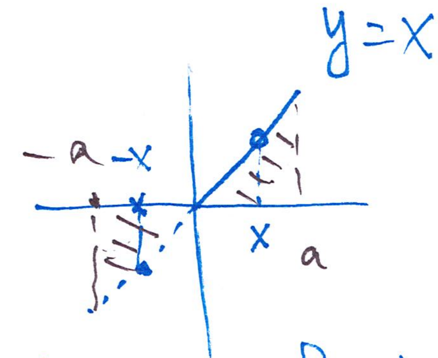


- symmetry

Integrals of Symmetric Functions $\quad f$ is continuous on $[-a, a]$

(a) f is even $\Longrightarrow \displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

(b) f is odd $\Longrightarrow \displaystyle\int_{-a}^{a} f(x)\,dx = 0$



$f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$u=-x$, $du=-dx$

$\displaystyle\int_a^0 f(-u)(-du) + \int_0^a f(x)\,dx$

$\displaystyle = -\int_a^0 -f(u)\,du = \int_a^0 f(u)\,dx = -\int_0^a f(u)\,du$

Ex. 10 $\int_{-2}^{2} (x^6+1)\,dx = 2\int_0^2 (x^6+1)\,dx$

$f(-x) = f(x)$

$= 2\left[\frac{1}{7}x^7 + x\right]_0^2 = 2\left[\frac{1}{7}\cdot 2^7 + 2\right]$

Ex. 11 $\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\,dx = \int_{-1}^{1} f(x)\,dx = 0$

$\tan x = \frac{\sin x}{\cos x}$

$\tan(-x) = \frac{\sin(-x)}{\cos(-x)}$

$= \frac{-\sin x}{\cos x} = -\tan x$

$f(-x) = \frac{-\tan x}{1+x^2+x^4} = -f(x)$

§3.8 Exponential Growth and Decay

Let $y(t)$ be the value of a quantity y at time t

$$\int \frac{1}{y}\frac{dy}{\cancel{dt}} = \int k\,dt$$

$$\ln y = kt + C$$

e   e

the solution

$$\boxed{\frac{dy}{dt} = ky}$$

ODE

law of natural growth if $k>0$

law of natural decay if $k<0$

$$y = C e^{kt}$$

$y(0)=C$

$$\boxed{y(t) = y(0)\,e^{kt}}$$

- population growth $P(t)$ denotes the size of a population at time t.

$$\frac{dP}{dt} = kP \qquad \text{or} \qquad \frac{1}{P}\frac{dP}{dP} = k$$

relative growth rate

Ex. 1 Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

$$P(t) = P(0)\, e^{kt}$$

$$P(0) = 2560, \quad P(10) = 3040$$

$$k = \frac{1}{P}\frac{dP}{dt}$$

$$P(10) = P(0)\, e^{10k}$$

$$3040 = 2560\, e^{10k}$$

$$e^{10k} = \frac{304\cancel{0}}{256\cancel{0}}, \quad 10k = \ln\frac{304}{256}$$

$$k = \frac{1}{10}\ln\frac{304}{256}$$

$$? = P(43) = P(0)\, e^{43k}$$

$$\cancel{? = P(70)} = 2560\, e^{43\cdot\frac{1}{10}\ln\frac{304}{256}}$$

$$\approx 5360$$

$$? = P(70) = 2560\, e^{70\cdot\frac{1}{10}\ln\frac{304}{256}} \approx 8524$$

- Radioactive Decay

$m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t.

$$\frac{dm}{dt} = k m \quad \text{with } k<0 \implies \boxed{m(t) = m_0 e^{kt}}$$

Ex. 2 The half-life of radium-266 is 1590 years.

(a) A sample of radium-266 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t years.

(b) Find the mass after 1000 years correct to the nearest milligram.

(c) When will the mass be reduced to 30 mg?

(a) $m_0 = 100$ mg, $m(1590) = \frac{1}{2} m_0 = \cancel{50 \text{ mg}} = m_0 e^{k \cdot 1590}$

$$\frac{1}{2} = e^{1590 k} \qquad 1590 k = \ln\frac{1}{2} = -\ln 2$$

$$\boxed{k = -\frac{\ln 2}{1590}}$$

$$m(t) = 100 e^{-\frac{\ln 2}{1590} t}$$

(b) $m(1000) = 100 e^{-\frac{\ln 2}{1590} \cdot 1000} \approx 65$ mg

(c) $\cancel{30 = m(t)}$ $30 = m(t) = 100 e^{-\frac{\ln 2}{1590} t}$ $\quad e^{-\frac{\ln 2}{1590} t} = 0.3$

$$-\frac{\ln 2}{1590} t = \ln \frac{3}{10} \qquad t = -\frac{(\ln 3 - \ln 10)}{\ln 2} 1590 \approx 2762$$