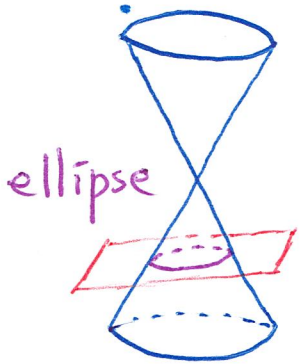
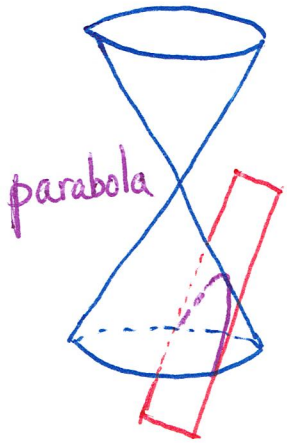
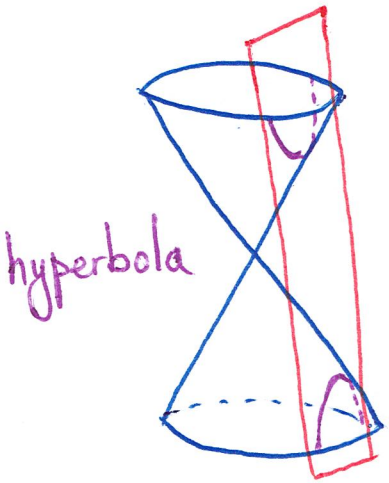
§10.5 Conic Sections

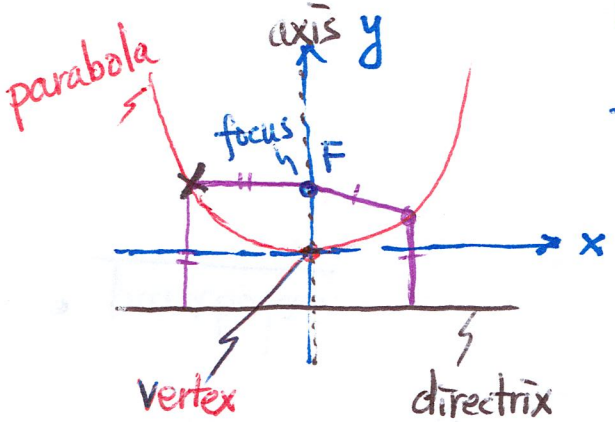
Conic sections or Conics are the intersections of cone and plane.







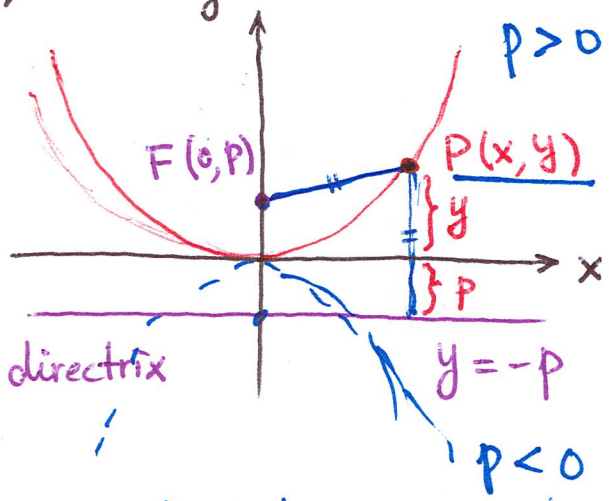
- **parabolas**



A parabola is the set of points in a plane that are equidistance from a fixed pt F (focus) and a fixed line (directrix)

• equation of parabola vertex at the origion

$a^2 - b^2 = (a+b)(a-b)$



$$|PF| = |y+p| \Rightarrow x^2 + (y-p)^2 = (y+p)^2$$

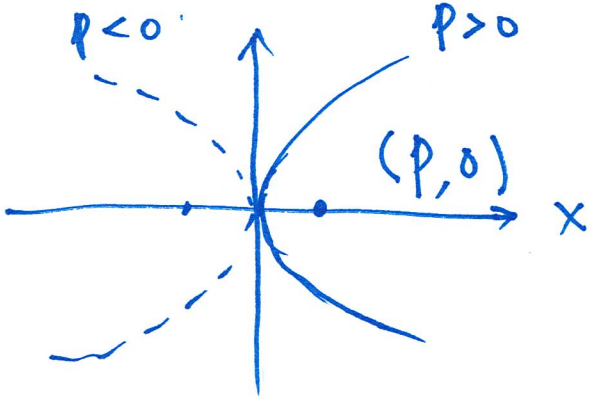
$$\sqrt{x^2+(y-p)^2} \qquad \Rightarrow x^2 = (y+p)^2 - (y-p)^2$$

$$= 2y \cdot 2p$$

$$\boxed{x^2 = 4py}$$

the equation of the parabola with focus $(0,p)$ and directrix $y=-p$

$$\boxed{x^2 = 4py}$$ or $y = ax^2$ with $a = \frac{1}{4p}$.



$$y^2 = 4px$$

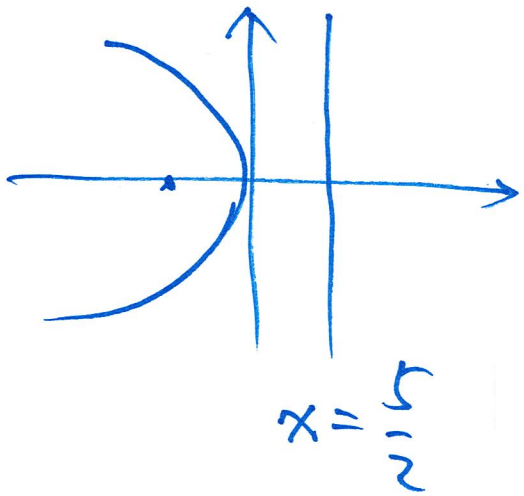
Ex. 1 Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

$y^2 = -10x = 4 \cdot \left(-\frac{5}{2}\right) x$

$p = -\frac{5}{2}$

focus $\left(-\frac{5}{2}, 0\right)$

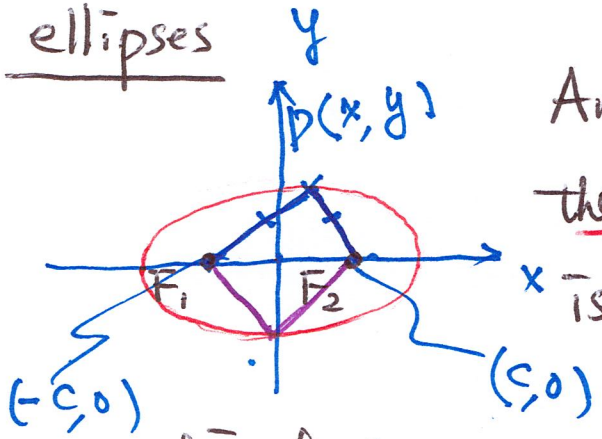
directrix $x = \frac{5}{2}$



$(a-b)^2 = a^2+b^2-2ac$

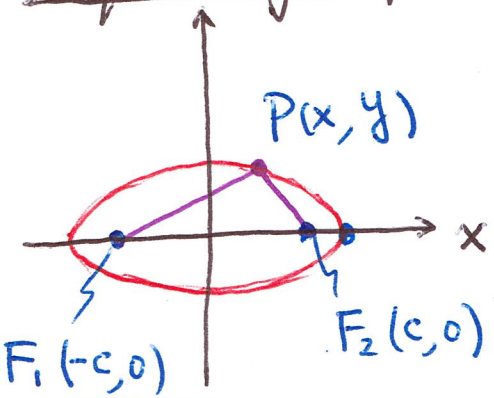
$a^2-b^2 = (a+b)(a-b)$

- **ellipses**



An ellipses is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant.

- **an equation of ellipses**



$$|PF_1| + |PF_2| = \text{constant} = 2a$$

$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$$

$$(x+c)^2+y^2 = 4a^2 + (x-c)^2+y^2 - 4a\sqrt{(x-c)^2+y^2}$$

$$2x\cdot 2c = 4a^2 - 4a\sqrt{(x-c)^2+y^2}$$

$$cx - a^2 = -a\sqrt{(x-c)^2+y^2}$$

$$(cx-a^2)^2 = a^2[(x-c)^2+y^2]$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $b^2 = a^2-c^2$

$$\left(\frac{c}{a}x - a\right)^2 = (x-c)^2+y^2$$

$$\left(\frac{c}{a}\right)^2x^2 + a^2 - 2cx = x^2+c^2 - 2cx+y^2$$

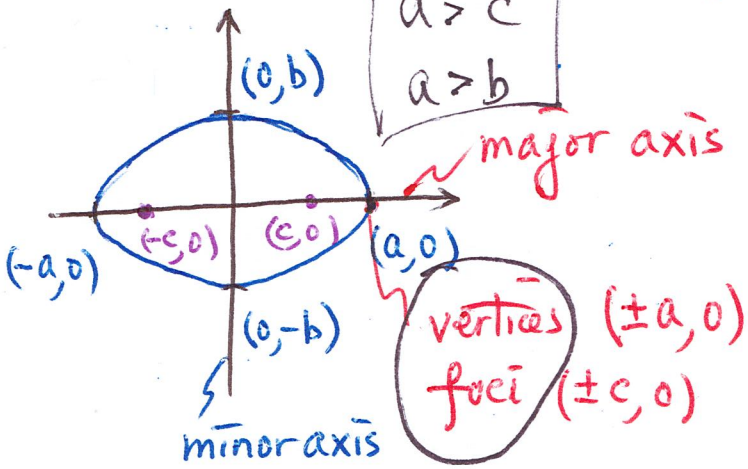
$$\frac{(a^2-c^2)}{a^2}x^2 + y^2 = \left[1-\left(\frac{c}{a}\right)^2\right]x^2 + y^2 = a^2-c^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$$

$c^2 = a^2-b^2$

$a > c$

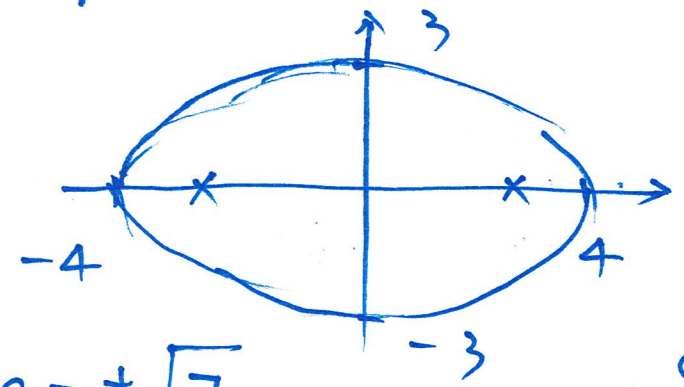
$a > b$



Ex. 2 Sketch the graph of $9x^2+16y^2=144$ and locate the foci. $(\pm\sqrt{7}, 0)$

$9 \cdot 16$

$$\frac{x^2}{4^2}+\frac{y^2}{3^2}=1$$



$y=0 \quad \frac{x^2}{4^2}=1 \Rightarrow x=\pm 4$

vertices $(\pm 4, 0)$

$x=0 \quad \frac{y^2}{3^2}=1 \Rightarrow y=\pm 3$

$c=?$ $\quad b^2=a^2-c^2 \Rightarrow c^2=a^2-b^2=16-9=7$

$c=\pm\sqrt{7}$

Ex. 3 Find an equation of the ellipses with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

$\Downarrow$ $\qquad a=3$

$c=2$

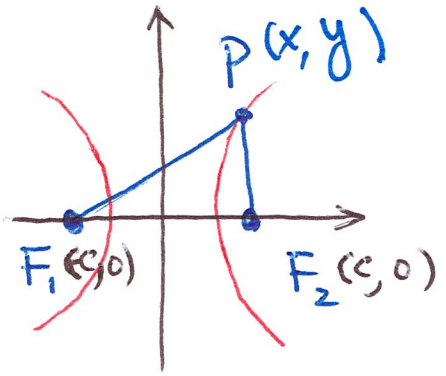
$b^2=a^2-c^2=9-4=5$

$\Rightarrow b=\sqrt{5}$

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

$$\frac{x^2}{5}+\frac{y^2}{9}=1$$

• hyperbolas



A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 is a constant.

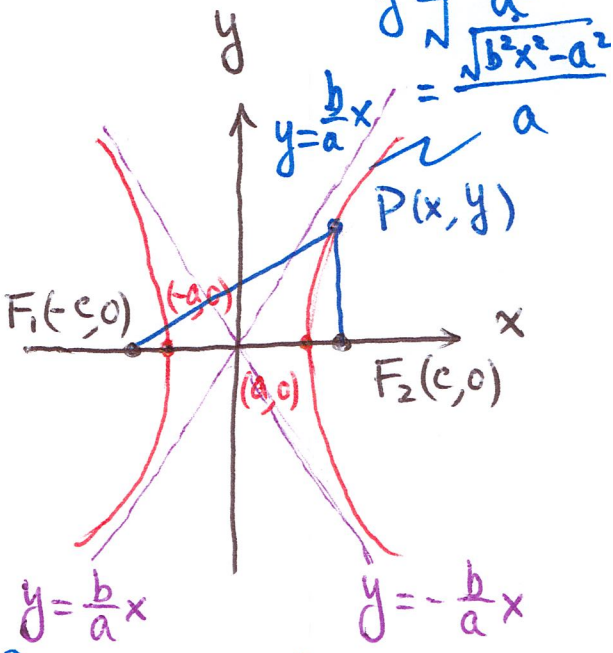
$(\pm c, 0)$ foci and $(\pm a, 0)$ vertices

the equation of hyperbola with the foci $(\pm c, 0)$

$$|PF_1| - |PF_2| = \pm 2a$$

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a$$

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \quad \text{with} \quad \boxed{c^2 = a^2 + b^2}$$



$$y = \sqrt{\frac{b^2x^2}{a^2} - 1} = \frac{\sqrt{b^2x^2 - a^2}}{a}$$

asymptotes $\quad y = \pm \frac{b}{a} x$

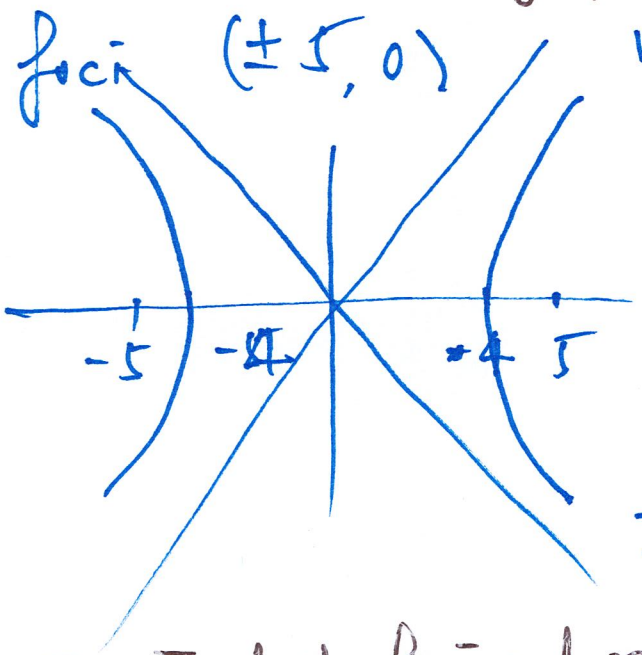
$$\lim_{x\to\infty}\left(y - \frac{b}{a}x\right) = \lim_{x\to\infty}\left(\frac{b}{a}\left(\sqrt{x^2 - \left(\frac{a}{b}\right)^2} - x\right)\right) = \frac{b}{a}\lim_{x\to\infty}\frac{\left(\sqrt{x^2 - \left(\frac{a}{b}\right)^2} - x\right)\left(\sqrt{x^2 - \left(\frac{a}{b}\right)^2} + x\right)}{\sqrt{x^2 - \left(\frac{a}{b}\right)^2} + x}$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$= \lim_{x\to\infty}\frac{b}{a}\frac{-\left(\frac{a}{b}\right)^2}{\sqrt{x^2 - \left(\frac{a}{b}\right)^2} + x} = 0$$

Ex. 4 Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

$\frac{9x^2 - 16y^2}{9 \cdot 16} = \frac{144}{9 \cdot 16}$



vertices $(\pm 4, 0)$

$\left(\frac{x}{4} + \frac{y}{3}\right)\left(\frac{x}{4} - \frac{y}{3}\right) = \frac{x^2}{16} - \frac{y^2}{9} = 1$

$\frac{x}{4} + \frac{y}{3} = 0 \Rightarrow y = -\frac{3}{4}x$

$\frac{x}{4} - \frac{y}{3} = 0 \Rightarrow y = \frac{3}{4}x$

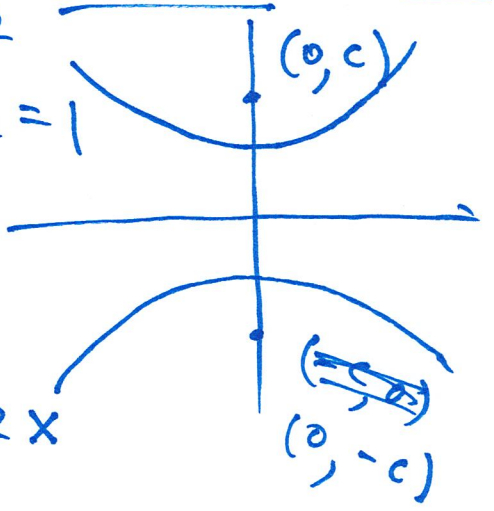
$a = 4, \quad b = 3$

$c^2 = a^2 + b^2 = 16 + 9 = 25$

$c = \pm 5$

Ex. 5 Find the foci and equation of the hyperbola with vertices $(0, \pm 1) = (0, \pm a)$ and asymptotes $y = 2x$.

$\left(\frac{y}{a} + \frac{x}{b}\right)\left(\frac{y}{a} - \frac{x}{b}\right) = \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$



$a = 1, \quad b = \frac{1}{2}$

$c^2 = a^2 + b^2 = 1 + \frac{1}{4} = \frac{5}{4}$

foci $\left(0, \pm \frac{\sqrt{5}}{2}\right)$

$\frac{y^2}{1} - \frac{x^2}{\frac{1}{4}} = 1$

$y = \frac{a}{b}x = 2x$

$1 = a = 2b \Rightarrow b = \frac{1}{2}$