Chapter 2 Limits and Derivatives (5 lectures)

§2.2 Limit of a Function

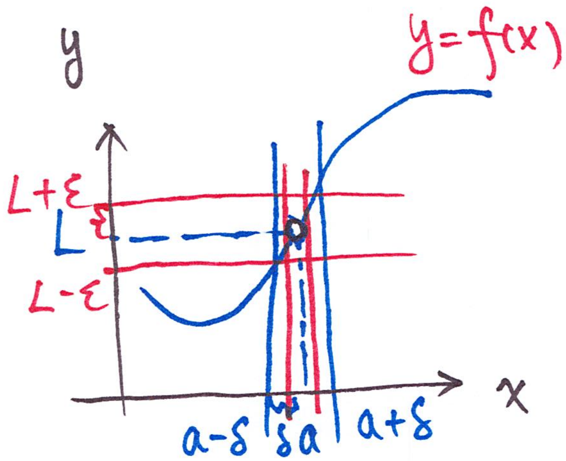
$\lim_{x \to a} f(x) = L$

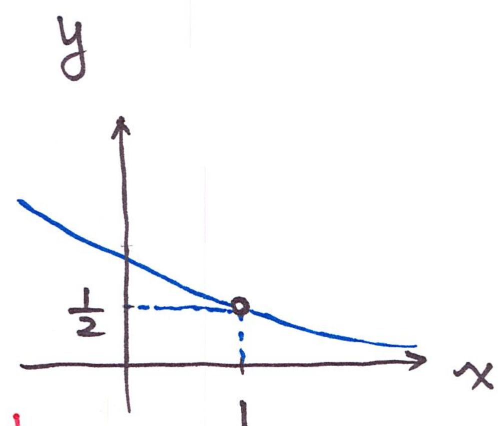
$\iff$ values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a, but $x \neq a$

$\iff$ $|f(x) - L|$ can be made arbitrarily small by taking $0 < |x-a|$ sufficiently small

$\iff$ $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, s.t.

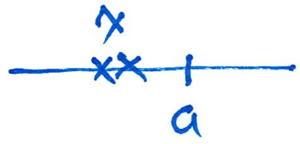
$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$





Ex. 1 $\lim_{x \to 1} \dfrac{x-1}{x^2-1} = \lim_{x \to 1} \dfrac{\cancel{x-1}}{(x+1)\cancel{(x-1)}} = \lim_{x \to 1} \dfrac{1}{x+1} = \dfrac{1}{2}$ $\qquad a^2 - b^2 = (a+b)(a-b)$

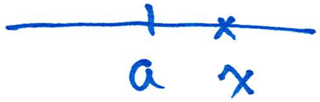
One-sided Limits



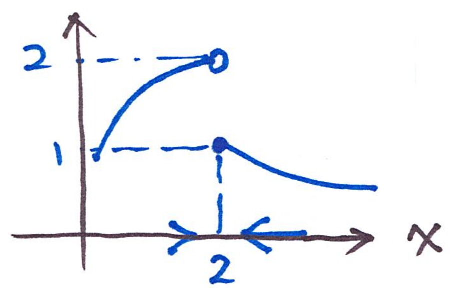
$$\lim_{x\to a^-} f(x) = L \iff \forall \varepsilon>0,\quad \exists \delta>0,\quad \text{s.t.}$$

$$0 < a-x < \delta \implies |f(x)-L| < \varepsilon$$

$$\lim_{x\to a^+} f(x) = L \iff \forall \varepsilon>0,\quad \exists \delta>0,\quad \text{s.t.}$$



$$0 < x-a < \delta \implies |f(x)-L| < \varepsilon$$



$$\lim_{x\to 2^-} f(x) = 2$$

$$\lim_{x\to 2^+} f(x) = 1$$

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

infinite Limits

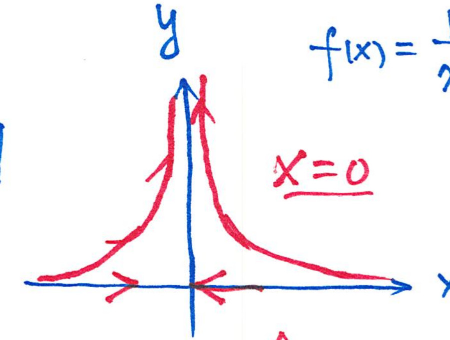
$\lim\limits_{x \to a} f(x) = +\infty \iff \forall N > 0, \exists \delta > 0$, s.t.

$0 < |x-a| < \delta \implies f(x) > N$

$\lim\limits_{x \to a} f(x) = -\infty \iff \forall N > 0, \exists \delta > 0$, s.t.

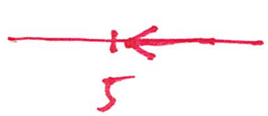
$0 < |x-a| < \delta \implies f(x) < -N$

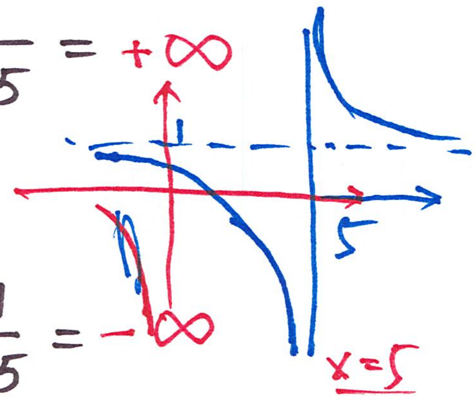
$f(x) = \frac{1}{x^2}$



$\lim\limits_{x \to 0^+} \frac{1}{x^2} = \infty$

Ex. $\lim\limits_{x \to 5^+} \frac{x+1}{x-5} = +\infty$





$\lim\limits_{x \to 5^-} \frac{x+1}{x-5} = -\infty$

$x-5<0$

$\lim\limits_{x \to \frac{\pi}{2}^-} \tan x = \lim\limits_{x \to \frac{\pi}{2}^-} \frac{\sin x}{\cos x} = +\infty$

$x \to \frac{\pi}{2}^+$ $\quad \lim\limits_{x \to \frac{\pi}{2}^+} \frac{\sin x}{\cos x} = -\infty$

$\lim\limits_{x \to -\frac{\pi}{2}} \tan x =$

Vertical Asymptote $x = a$ is a vertical asymptote of the curve $y = f(x)$

$\iff \lim\limits_{x \to a} f(x) = \pm\infty$ or $\lim\limits_{x \to a^{\pm}} f(x) = \pm\infty$

§2.3 Calculating Limits Using the Limit Laws

Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

(1) $\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$; (2) $\lim_{x\to a}\left[c f(x)\right] = c \lim_{x\to a} f(x)$;

(3) $\lim_{x\to a}\left[f(x)\, g(x)\right] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$; (4) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ - if $\lim_{x\to a} g(x) \neq 0$

(5) $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$; (6) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ for even n $\lim_{x\to a} f(x) > 0$

(7) $\lim_{x\to a} c = c$; (8) $\lim_{x\to a} x = a$; (9) $\lim_{x\to a} x^n = a^n$

$$\boxed{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n = P_n(x)}$$

(10) $\lim_{x\to a} x^{\frac{1}{n}} = a^{\frac{1}{n}}$ $n > 0$; for even n, $a > 0$.

$$\frac{P_n(x)}{q_m(x)}$$

$$\lim_{x\to a} P_n(x) = P_n(a)$$

Examples

$$\lim_{x\to 5}\left(2x^2-3x+4\right) = 2\cdot 5^2 - 3\cdot 5 + 4$$

$$= 2\lim_{x\to 5}\left(\cancel{2}x^2\right) - 3\lim_{x\to 5}\left(\cancel{3}x\right) + \lim_{x\to 5} 4$$

$$= 2\cdot 5^2 - 3\cdot 5 + 4$$

$$\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x} = \frac{\lim\left(x^3+2x^2-1\right)}{\lim\left(5-3x\right)} = \frac{\lim x^3 + 2\lim x^2 - \lim 1}{\lim 5 - 3\lim x}$$

$$= \frac{(-2)^3+2(-2)^2-1}{5-3\cdot(-2)} = \frac{-8+8-1}{5+6} = -\frac{1}{11}$$

Direct Substitution Property Assume that $f(x)$ is a polynomial or rational function

$$\lim_{x\to a} f(x) = f(a)$$

- $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$ if $f(x)=g(x)$ when $x\neq a$ $\longrightarrow$ $a^2-b^2=(a+b)(a-b)$

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

$$a^3+a^2b+ab^2-ba^2-ab^2-b^3$$

examples

$$\lim_{x\to 1}\frac{x^2-1}{x-1} \;\left(\frac{0}{0}\right) = \lim_{x\to 1}\frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1}(x+1) = 2$$

$$\lim_{x\to 4}\frac{x^2-4x}{x^2-3x-4} \;\left(\frac{0}{0}\right) = \lim_{x\to 4}\frac{x\,\cancel{(x-4)}}{\cancel{(x-4)}(x+1)} = \lim_{x\to 4}\frac{x}{x+1} = \frac{4}{5}$$

$$\begin{array}{cc} x & -4 \\ x & 1 \end{array}$$

$$(x^3)' = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} \;\left(\frac{0}{0}\right) = \lim_{h\to 0}\frac{\cancel{\left((x+h)-x\right)}\left((x+h)^2+(x+h)x+x^2\right)}{\cancel{h}} = x^2+x^2+x^2 = 3x^2$$

$$\lim_{u\to 2}\frac{\sqrt{4u+1}-3}{u-2} \;\left(\frac{0}{0}\right)\cdot\frac{\sqrt{4u+1}+3}{\sqrt{4u+1}+3} = \lim\frac{(4u+1)-9}{\cancel{(u-2)}\left[\sqrt{4u+1}+3\right]}, \quad (4u+1)-9 = 4\cancel{(u-2)}$$

$$= \frac{4}{\sqrt{8+1}+3} = \frac{4}{6} = \frac{2}{3}$$