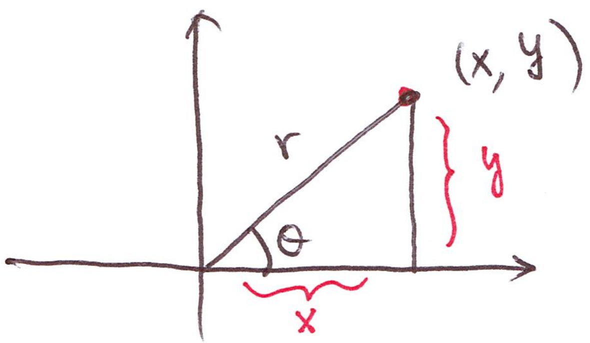
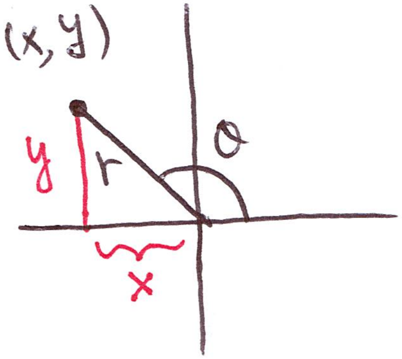
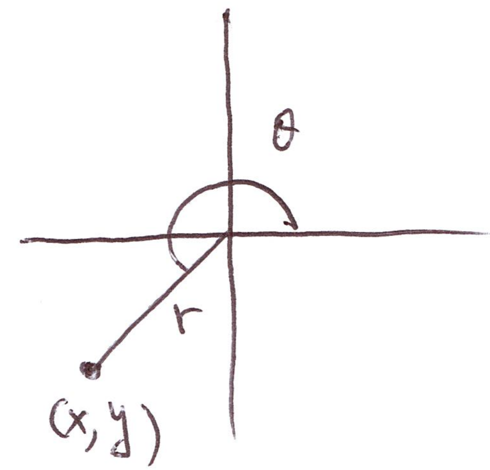
Homework 2

#15 $\cos x = -\frac{1}{7}$, $\pi < x < \frac{3\pi}{2}$

Find $\sin x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$







$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r} = -\frac{1}{7} \Rightarrow x = -1,\ r = 7$$

$$\Rightarrow y = \sqrt{r^2 - x^2} = \sqrt{49 - 1}$$

$$= \sqrt{48}$$

$$\lim_{x\to 1^-} \frac{x^2-1}{x-1} \qquad a^2-b^2=(a+b)(a-b) \qquad \lim_{x\to 1^-} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$$

$a=x, \; b=1$

$$\frac{0}{0} \qquad \frac{\infty}{\infty}$$

$$\lim_{x\to 4^-} \frac{x^2-4x}{x^2-3x-4} \qquad \text{factorization} \qquad \lim_{x\to 4^-} \frac{x\cancel{(x-4)}}{\cancel{(x-4)}(x+1)}$$

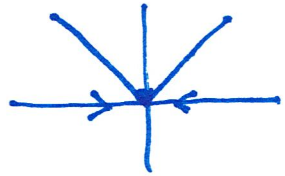
$$\lim_{h\to 0^-} \frac{(x+h)^3-x^3}{h} \qquad a^3-b^3=(a-b)(a^2+ab+b^2) \qquad \lim_{h\to 0^-} \frac{\cancel{(x+h-x)}\left((x+h)^2+(x+h)x+x^2\right)}{\cancel{h}}$$

$a=x+h, \; b=x$

$$\lim_{u\to \cancel{0}\,2^-} \frac{\sqrt{4u+1}-3}{u-2} \cdot \frac{\sqrt{4u+1}+3}{\sqrt{4u+1}+3} = \lim_{u\to 0^-} \frac{(4u+1)-3^2}{\cancel{(u-2)}\left(\sqrt{4u+1}+3\right)} = 4\cancel{(u-2)}$$

- $\lim\limits_{x\to a} f(x) = L \iff \underline{\lim\limits_{x\to a^-} f(x)} = L = \underline{\lim\limits_{x\to a^+} f(x)}$

Ex. 7 Show that $\lim\limits_{x\to 0} |x| = \underline{0}$



$$|x| = \begin{cases} -x, & x < 0 \\ x & \underline{x \geq 0} \end{cases}$$

$$= \lim_{x\to 0} \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$\lim\limits_{x\to 0^-} |x| = 0$, $\lim\limits_{x\to 0^+} |x| = \lim\limits_{x\to 0^+} x = 0$

Ex. 8 Show that $\lim\limits_{x\to 0} \frac{|x|}{x}$ does not exist.

$$\lim_{x\to 0^-} \frac{|x|}{x} \overset{x<0}{=} \lim_{x\to 0^-} \frac{-x}{x} = -1 \neq \lim_{x\to 0^+} \frac{|x|}{x} \overset{x>0}{=} \lim_{x\to 0^+} \frac{x}{x} = 1$$

Ex. 9

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

$\lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$

$\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^-} (8-2x) = 8 - 2\cdot 4 = 0$

$\lim\limits_{x\to 4} f(x) = 0$

- $f(x) \le g(x)$ for all $x \neq a$ near a $\Longrightarrow$ $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$

L

- (The Squeeze Thrm) $f(x) \le g(x) \le h(x)$ $\forall\, x \neq a$ but near a

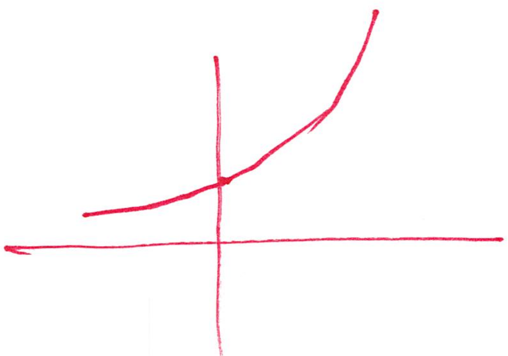
$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x) \Longrightarrow \lim_{x\to a} g(x) = L$$

example 11 show that $\underbrace{\lim\limits_{x\to 0^+} \sqrt{x}\, e^{\sin\left(\frac{\pi}{x}\right)}}_{\downarrow\; 0} = 0.$

$$-1 \le \sin\frac{\pi}{x} \le 1.$$

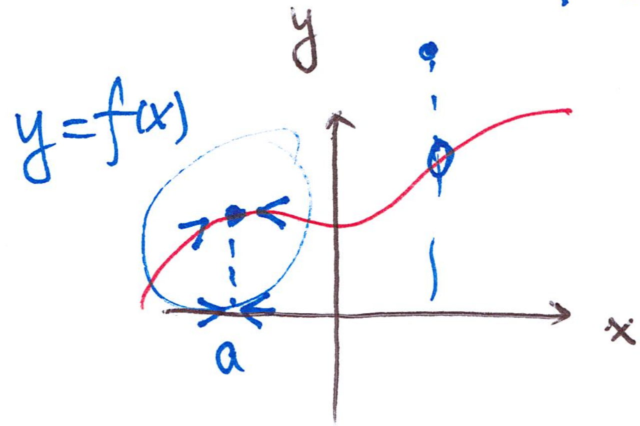
$$e^{-1} \le e^{\sin\frac{\pi}{x}} \le e$$

$$\underset{\downarrow\; 0}{\sqrt{x}\, e^{-1}} \le \sqrt{x}\, e^{\sin\frac{\pi}{x}} \le \underset{\downarrow\; 0}{\sqrt{x}\, e}$$



§2.5 Continuity

$C = \{(x, f(x)) \mid x \dots\}$



Def. $f(x)$ is continuous at $x = a$

$$\Longleftrightarrow \lim_{x \to a} f(x) = f(a)$$

$\Updownarrow$

(1) $\lim_{x \to a^-} f(x)$ exists

(2) $\lim_{x \to a^+} f(x)$ exists

(3) $\lim_{x \to a} f(x) = f(a)$

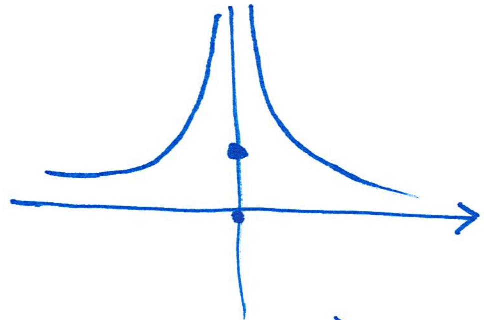
$$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right)$$

Examples 2 where are each of the following functions discontinuous? why?

(a) $f(x) = \dfrac{x^2-x-2}{x-2}$ $\qquad x-2=0 \qquad$ $f(x)$ is not cont. at $x=2$

$a \neq 2 \qquad \lim\limits_{x\to a} \dfrac{x^2-x-2}{x-2} = \dfrac{a^2-a-2}{a-2} = f(a)$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



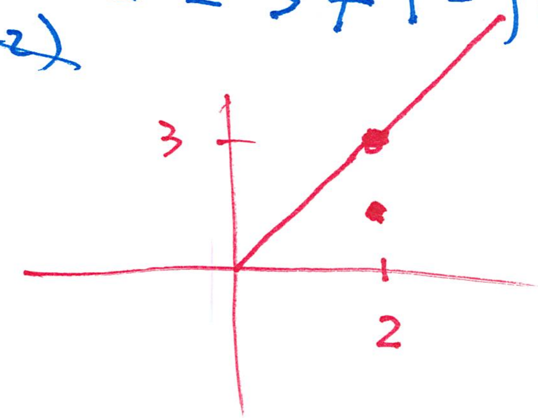
$\lim\limits_{x\to 0^-} \dfrac{1}{x^2} = \infty$

$\lim\limits_{x\to 0^+} \dfrac{1}{x^2} = \infty$

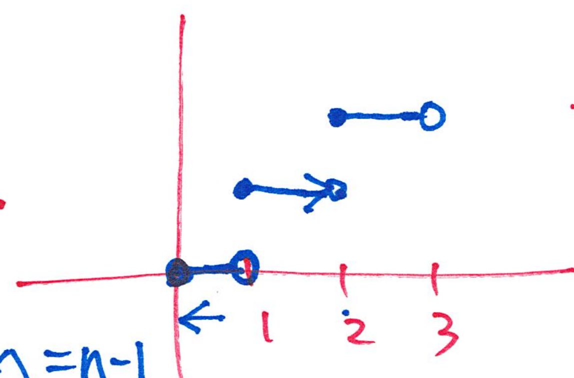
$f(0) = 1$

(c) $f(x) = \begin{cases} \dfrac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 1 \quad 3 & \text{if } x = 2 \end{cases}$

$\lim\limits_{x\to 2} \dfrac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}} = 3 \neq 1 = f(2)$



(d) $f(x) = [\![x]\!] = n$ if $n \leq x < n+1$

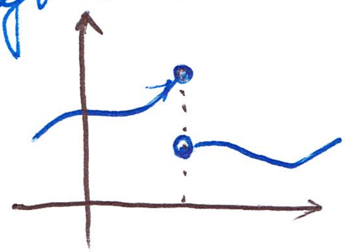


$\lim\limits_{x\to n^+} f(x) = n \neq \lim\limits_{x\to n^-} f(x) = n-1$

Def. $f(x)$ is continuous from the right at $x=a \iff \lim_{x\to a^+} f(x) = f(a)$

$f(x)$ is continuous from the left at $x=a \iff \lim_{x\to a^-} f(x) = f(a)$

left cont



right discont.

Ex. 3 $f(x) = [\![x]\!]$ is continuous from right but left at integers $x=n$.

Def. $f(x)$ is continuous on an interval $(a, b]$ $\quad (a,b)$, $[a,b)$, $[a,b]$

$\iff$ (1) $f(x)$ is continuous at every $x \in (a, b)$

(2) $f(x)$ is continuous at $x = b$ from the left.

Ex. 4 Show that the function $y = f(x) = 1 - \sqrt{1-x^2}$ is continuous on $[-1, 1]$

$$y = 1 - \sqrt{1-x^2}$$

$$y - 1 = -\sqrt{1-x^2}$$

$$(y-1)^2 = 1 - x^2$$

(1) ~~f(x)~~ f is cont. on $(-1, 1)$

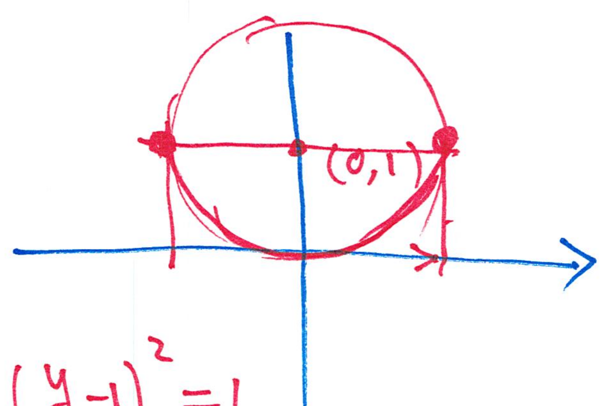
for every $a \in (-1, 1)$

$$\lim_{x \to a} f(x) \overset{?}{=} f(a)$$

$$\lim_{x \to a} \left(1 - \sqrt{1-x^2}\right) = 1 - \sqrt{1-a^2} = f(a)$$

(2) $x = -1$ $\quad \lim_{x \to -1^+} f(x) \overset{?}{=} f(-1) = 1 - \sqrt{1-(-1)^2} = 1$

(3) $x = 1$ $\quad \lim_{x \to 1^-} f(x) \overset{?}{=} f(1)$



$$x^2 + (y-1)^2 = 1$$

Theorem Assume that $f(x)$ and $g(x)$ are continuous at $x = a$

$\Longrightarrow$ $f(x) \pm g(x)$, $f(x) \times g(x)$, $f(x) / g(x)$ are continuous at $x = a$ (for $\frac{f}{g}$, if $g(a) \neq 0$)

$c f(x)$ is continuous at $x = a$

Proof of $f + g$ is cont.

$$\lim_{x \to a} \left(f(x) + g(x) \right) = f(a) + g(a)$$

Theorem The following types of functions are continuous at every number in their domain

(1) polynomials, (2) rational functions, (3) root functions, (4) trigonometric functions

(5) inverse trigonometric functions, (6) exponential and logarithmic functions

Theorem

$$\lim_{x \to a} f\left(g(x) \right) = f\left(\lim_{x \to a} g(x) \right) = f\left(g(a) \right)$$

Ex. (#24, P125) How to define $f(2)$ in order to make f is continuous at 2?

$$f(x) = \frac{x^3 - 8}{x^2 - 4} \quad = x^3 - 2^3$$

$$f(2) = \lim_{x \to 2} f(x) = 3$$

$$= \frac{\cancel{(x-2)}(x^2 + 2x + 2^2)}{(x+2)\cancel{(x-2)}} \longrightarrow \frac{4+4+4}{4} = 3$$

Ex. (#35, P125) Use continuity to evaluate the limit

$$\lim_{x \to 2} x\sqrt{20 - x^2}$$

Ex. (#45, P125) For what value of the constant c is f continuous on $(-\infty, +\infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$