Ex. (#24, P125) How to define f(2) in order to make f is continuous at 2?

$$f(x) = \frac{x^3-8}{x^2-4} \quad (x^3 - 8 = x^3-2^3) \qquad f(2) = \lim_{x\to 2} f(x) = 3$$

$$= \frac{\cancel{(x-2)}(x^2+2x+2^2)}{(x+2)\cancel{(x-2)}} \longrightarrow \frac{4+4+4}{4} = 3$$

Ex. (#35, P125) Use continuity to evaluate the limit

$$\lim_{x\to 2} x\sqrt{20-x^2} = 2\cdot\sqrt{20-2^2} = 2\sqrt{20-4} = 2\cdot 4 = 8$$

$$20 - x^2 \geq 0 \iff x^2 \leq 20 \iff |x| \leq \sqrt{20} \iff -\sqrt{20} \leq x \leq \sqrt{20}$$

Ex. (#45, P125) For what value of the constant c is f continuous on $(-\infty, +\infty)$?

$$f(x) = \begin{cases} cx^2+2x & \text{if } x<2 \\ x^3-cx & \text{if } x\geq 2 \end{cases}$$

$x = 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (cx^2+2x) = c\cdot 2^2 + 2\cdot 2 = \cancel{4}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^3-cx) = 8-2c = 4c+4$$

$$6c = 4 \Rightarrow c = \frac{2}{3}$$

$$f(2) = 8 - \frac{2}{3}\cdot 2 = \frac{20}{3}$$

#46 Find a, b

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x < 2 \\ ax^2 - bx + 3, & 2 \le x < 3 \\ 2x - a + b, & x \ge 3 \end{cases}$$

$x = 2$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{(x+2)(x-2)}{x-2} = 4$$

$$\lim_{x \to 2^+} f(x) = 4a - 2b + 3$$

$$\Rightarrow \quad 4a - 2b + 3 = 4$$

$$\begin{cases} 4a - 2b = 1 \\ \cancel{8}^{10}a - 4b = 3 \end{cases}$$

$x = 3$

$$\lim_{x \to 3^-} f(x) = 9a - 3b + 3$$

$$\lim_{x \to 3^+} f(x) = 6 - a + b$$

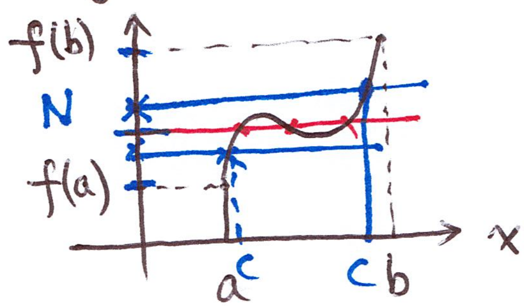
$$\Rightarrow \quad 9a - 3b + 3 = 6 - a + b$$

$$10\cancel{8}a - 4b = 3$$

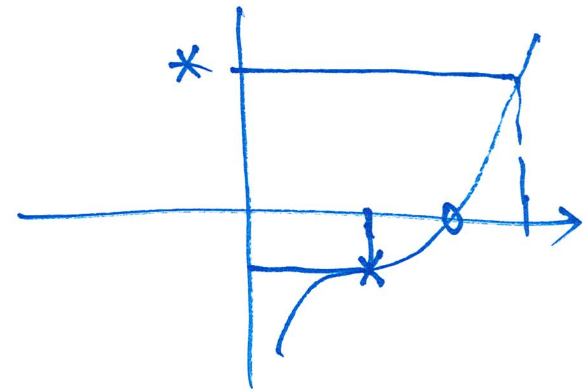
Intermediate Value Thrm

Assume that f is continuous on $[a,b]$, N is between $f(a)$ and $f(b)$, and $f(a) \neq f(b)$

$\Longrightarrow$ $\exists\, c \in (a,b)$ such that $f(c) = N$.



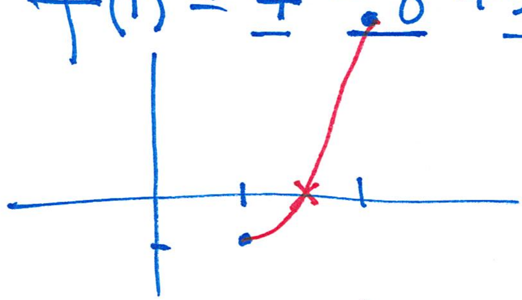
$\boxed{f(x) = 0}$



$f(x) = 0$

Ex. 10 Show that the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has a root between 1 and 2.

$f(1) = 4 - 6 + 3 - 2 = -1 < 0$, $f(2) = 4 \cdot 8 - 6 \cdot 4 + 6 - 2$
$= 32 - 24 + 6 - 2 = 12 > 0$



$\boxed{f(x) = \ln x - x + \sqrt{x} = 0}$

Ex. (#54, P126) $\ln x = x - \sqrt{x}$, $(2,3)$

$f(2) = \ln 2 - 2 + \sqrt{2} < 0$

$f(3) = \ln 3 - 3 + \sqrt{3} > 0$

§2.6 Limits at Infinity; Horizontal Asymptotes

$|f(x) - L|$ is arbitrarily small

~~if~~ when x is suff. large

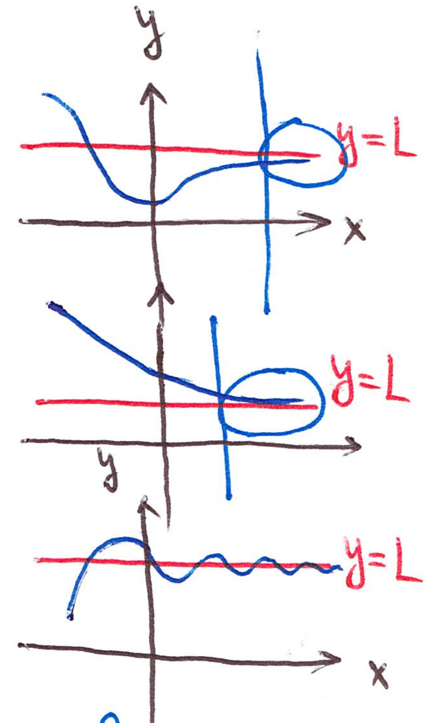
$$\lim_{x \to +\infty} f(x) = L$$

$\forall \varepsilon > 0, \; \exists N > 0$ s.t.

$$\lim_{x \to -\infty} f(x) = L$$

$x > N \implies |f(x) - L| < \varepsilon$

$x < -N$



$\boxed{y = L \text{ — horizontal asymptote}} \iff \lim_{x \to \pm\infty} f(x) = L$

Examples $\quad \dfrac{\infty}{\infty}$

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$$\lim_{\substack{x \to +\infty \\ -\infty}} \frac{1}{x^r} = 0 \quad \text{for } r > 0$$

$$\lim_{x \to +\infty} \frac{x^2 - 1}{x^2 + 1} = \lim_{x \to +\infty} \frac{\frac{x^2-1}{x^2}}{\frac{x^2+1}{x^2}} = \lim_{x \to +\infty} \frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x^2}} = 1 \qquad \lim_{x \to \infty} \frac{1}{x} = 0$$

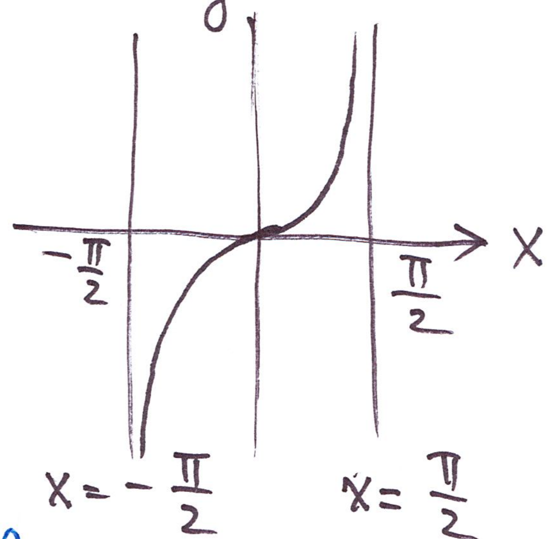
$$= \lim_{x \to +\infty} \frac{2x}{2x} = 1$$

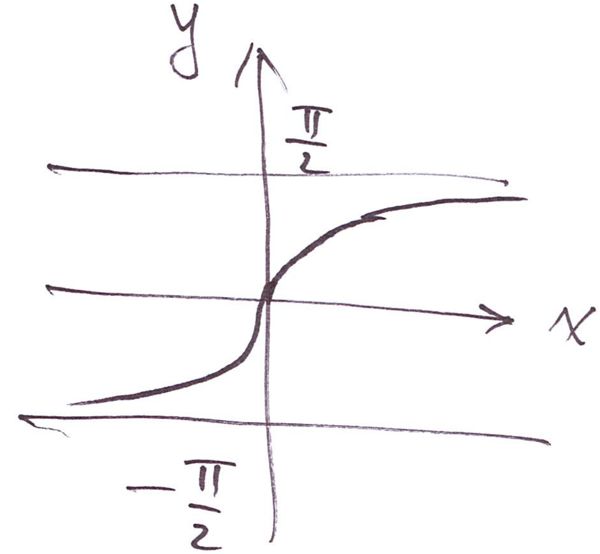
$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2},$$

$$\lim_{x\to+\infty} \tan^{-1} x = \frac{\pi}{2}$$





$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \quad \text{for } r>0$$

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0 \quad \text{for } r>0 \text{ and } x^r \text{ is defined.}$$

Examples

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2+4x+1} = \lim_{x\to\infty} \frac{\frac{3x^2-x-2}{x^2}}{\frac{5x^2+4x+1}{x^2}} = \lim \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} = \frac{3}{5}$$

$$= \frac{3}{5}$$

Ex. 4 Find the horizontal and vertical asymptotes of the graph of $f(x)=\frac{\sqrt{2x^2+1}}{3x-5}$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{3-\frac{5}{x}} = \sqrt{2+\frac{1}{x^2}} = \boxed{\frac{\sqrt{2}}{3} = y}$$

$$\frac{\sqrt{2}\,x}{3x} = \frac{\sqrt{2}}{3}$$

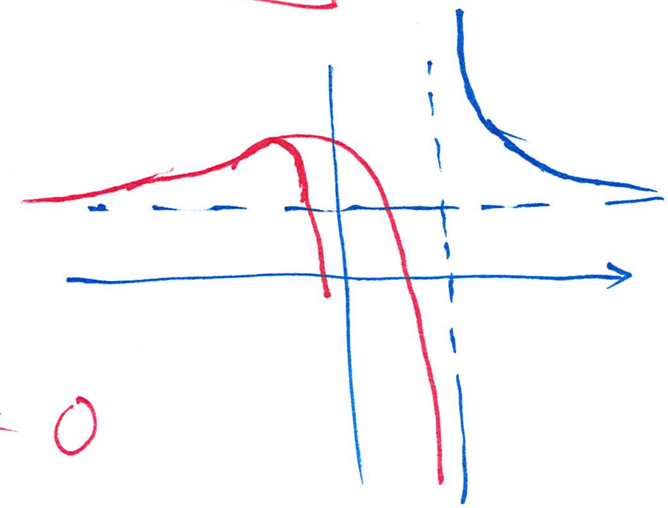
$$\lim_{x\to-\infty} f(x) = \frac{\sqrt{2}}{3}$$

$$3x-5=0 \Longrightarrow \boxed{x=\frac{5}{3}}$$

$$\lim_{x\to\frac{5}{3}^-} f(x) = -\infty$$

$$\boxed{a^2-b^2=(a+b)(a-b)} \quad \infty-\infty$$

$$\lim_{x\to\frac{5}{3}^+} f(x) = +\infty$$



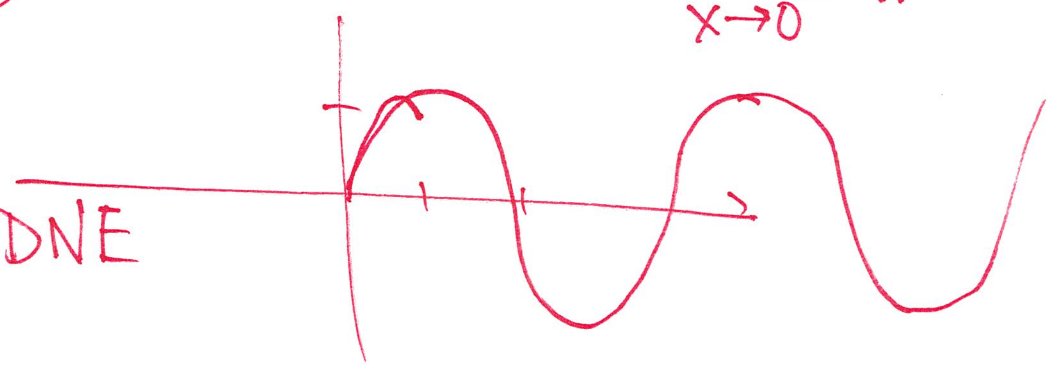
Ex. 5 $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$

$$=\lim \frac{\sqrt{x^2+1}-x}{1}\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim_{x\to\infty}\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = 0$$

Ex. 7 $\lim_{x\to0^-} e^{\frac{1}{x}} = 0$

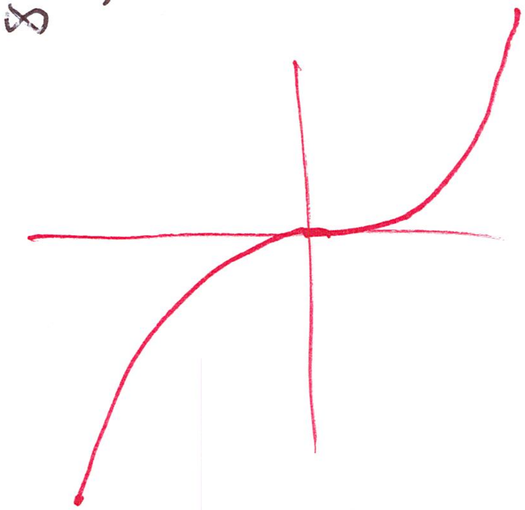
$$\lim_{x\to0^-}\frac{1}{x} = -\infty$$

Ex. 8 $\lim_{x\to\infty} \sin x$ DNE



- <u>Infinite Limits at Infinity</u>

$$\lim_{x \to +\infty} f(x) = \pm\infty, \quad \lim_{x \to -\infty} f(x) = \pm\infty$$



$x^3 = y$

$$\frac{0}{0}, \frac{\infty}{\infty}$$

$$\Uparrow$$

$$\infty - \infty$$

<u>Ex. 9</u> $\lim_{x \to \infty} x^3 = \infty$

$\lim_{x \to -\infty} x^3 = -\infty$

<u>Ex. 10</u> $\underline{\lim_{x \to \infty} (x^2 - \underline{x})} = \lim \underline{x}\,(\underline{\underline{x-1}}) = \infty$

<u>Ex. 11</u> $\lim_{x \to \infty} \dfrac{x^2 + x}{3 - x} =$