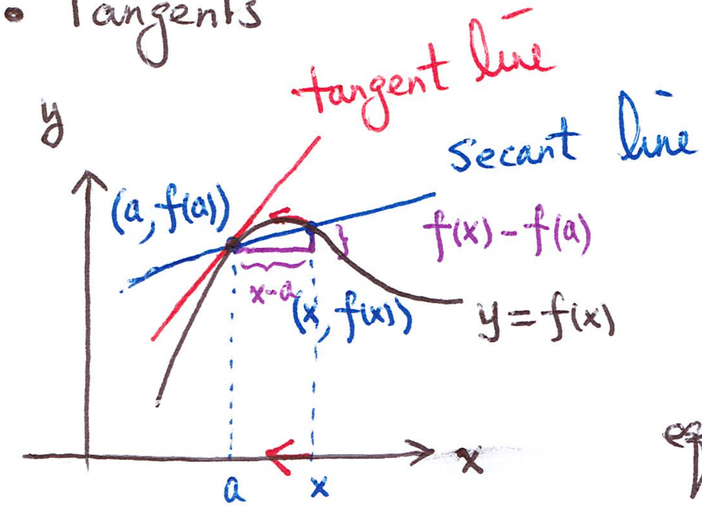
§2.7 Derivatives and Rate of Change

- Tangents



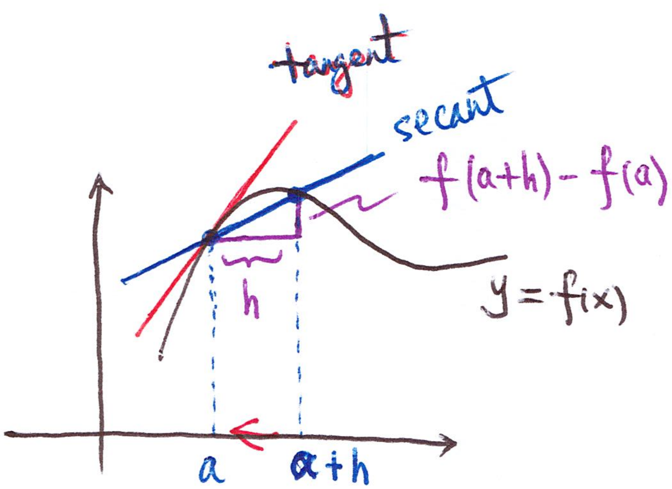
the slope of the secant line $= \dfrac{f(x)-f(a)}{x-a}$

the slope of the tangent line $= \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a} =$

equation of tangent line passing through $(a, f(a))$

$$m = \frac{y - f(a)}{x-a}$$

$$y = f(a) + m(x-a)$$



$h \to 0$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Ex. 1 Find an equation of the tangent line to $y = x^2$ at pt $(1, 1)$.

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = 2$$

$a = 1$

$y = f(a) = 1^2 = 1$

$$y = 1 + 2(x - 1)$$

$a^2 - b^2 = (a+b)(a-b)$

Ex. 2 Find an equation of the tangent line to the hyperbola $y = \frac{1}{x}$ at $(3, 1)$.

$a = 3$, $f(a) = 1$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$$

$$\frac{1}{3+h} - \frac{1}{3} = \frac{3 - (3+h)}{3(3+h)}$$

$$= \lim_{h \to 0} \frac{\frac{-\cancel{h}}{3(3+h)}}{\cancel{h}} = \lim_{h \to 0} -\frac{1}{3(3+h)} = -\frac{1}{9}$$

$$y = 1 - \frac{1}{9}(x - 3)$$

- Velocities

$s = f(t)$ the position function of a moving object along a straight line

$$\xrightarrow[a \quad a+h]{} t$$

average velocity at $[a, a+h]$ $= \dfrac{\text{displacement}}{\text{time}} = \dfrac{f(a+h) - f(a)}{h}$

instantaneous velocity at $t = a$ $= \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = \underline{v(a)}$

Ex. 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Galileo's Law

$s(t) = 4.9t^2$ m

$$v(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = \lim_{h \to 0} \frac{4.9(t+h)^2 - 4.9t^2}{h}$$

$$= 4.9 \lim_{h \to 0} \frac{(2t+h)\cancel{h}}{\cancel{h}} = 9.8t$$

$v(5) = 9.8 \times 5$
$= 49$

t^*, $v(t^*) = ?$ $\quad s(t^*) = 4.9\, {t^*}^2 = 450$

$$t^* = \sqrt{\frac{450}{4.9}}$$

$$v(t^*) = 9.8\, t^* = \sqrt{\frac{450}{4.9}} \times 9.8$$

• Derivatives • Rates of Change

the derivative of a function f at $x=a$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

Ex. 4 & 5 $y = f(x) = x^2 - 8x + 9$

(a) $f'(a) = ?$ (b) the eq. of the tangent line at $(3,-6)$?

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{(x^2-8x+\cancel{9}) - (a^2-8a+\cancel{9})}{x-a}$$

$$= \lim_{x \to a} \frac{(x^2-a^2) - 8(x-a)}{\cancel{x-a}} \qquad = (x+a)(x-a) - 8(x-a)$$

$$= [x+a-8]\,\cancel{(x-a)}$$

$$= \lim_{x \to a} [x+a-8] = 2a-8 \qquad f'(3) = 6-8 = -2$$

$$y = -6 - 2(x-3)$$

examples Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

#37 $\lim_{h\to 0} \frac{\boxed{\sqrt{9+h}-3 = f(a+h)-f(a)}}{h} = f'(a)$

$a = 9$

$f(a+h) = \sqrt{9+h}$

$f(x) = \sqrt{\cancel{9+h}\, x}$, $f(9) = \sqrt{9} = 3$

#39 $\lim_{x\to 2} \frac{x^6-64}{x-2}$ $= \underset{\underset{x^6}{\parallel}}{f(x)} - \underset{\underset{2^6}{\parallel}}{f(2)}$

$a=2$

#40 $\lim_{x\to \frac{1}{4}} \frac{\frac{1}{x}-4}{x-\frac{1}{4}}$ $= \underset{\underset{\frac{1}{x}}{\parallel}}{f(x)} - \underset{\underset{\frac{1}{\frac{1}{4}} = 4.}{\parallel}}{f(\tfrac{1}{4})}$

$a = \frac{1}{4}$

$\cancel{f(x) = \cos(a+}$

#41 $\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h}$ $= \underset{\underset{\cos(\pi+h)}{\parallel}}{f(a+h)} - f(a)$

$h = (a+h) - h$

$a = \pi$

$\cancel{f(a+x) = \cos(a+x}$

$f(\pi+h) = \cos(\pi+h)$

$f(x) = \cos x$

$\cos\pi = -1$

#42 $\lim_{\theta\to \frac{\pi}{6}} \frac{\sin\theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$ $= \underset{\underset{\sin\theta}{\parallel}}{f(\theta)} - \underset{\underset{\sin\frac{\pi}{6} = \frac{1}{2}}{\parallel}}{f(\tfrac{\pi}{6})}$

$a = \frac{\pi}{6}$

#37 $\lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h} = f'(9)$, $\underline{\underline{f(x) = \sqrt{x}}}$

Given

$$\lim_{h \to 0} \frac{\sqrt{9+h} - 3}{2h} =$$

and $f'(9) =$

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given

$f'(3) = 1$

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{9h} = ?$$

$$= \frac{1}{9} \underline{\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}} = \frac{1}{9} f'(3) = \frac{1}{9}$$