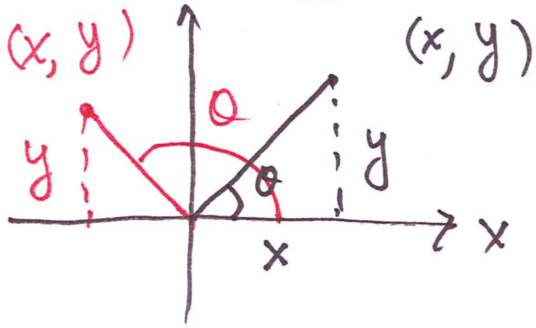
(1) $\sec\theta = -1.5$ and $\frac{\pi}{2} < \theta < \pi$, compute $\sin\theta$, ...

$= -\frac{3}{2}$



$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}$$

$$\cos\theta = -\frac{2}{3} \Rightarrow x = -2, \quad r = 3 \Longrightarrow y = \sqrt{3^2-2^2}$$

(2) Prove that

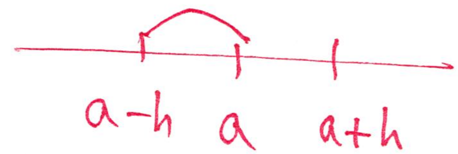
$$\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta = \sin^2\theta\left[\frac{1}{\cos^2\theta} - 1\right] = \sin^2\theta\,\frac{1-\cos^2\theta}{\cos^2\theta}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a} = \lim_{h \to 0} \frac{f(a) - f(a-h)}{a - (a-h)}$$

$$\lim_{h \to 0} \frac{f(\underline{a+h}) - f(\underline{a-h})}{2h} \cdot 2 = 2 f'(a)$$

$$(a+h) - (a-h) = 2h$$



$$\lim_{h \to 0} \frac{f(a) - f(a-h)}{h} = f'(a)$$

$$\lim_{h \to 0} \frac{f(\underline{a+h}) - f(\underline{a-2h})}{3h} \cdot 3 = 3 f'(a)$$

$$\lim_{h \to 0} \frac{f(\underline{a+3h}) - f(\underline{a-4h})}{7h} \, 7 = 7 f'(a)$$

§2.8 The Derivative as a Function

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Ex. 1 See Fig. 2 on P153.

Examples compute $f'(x)$ for (a) $f(x) = x^3 - x$, (b) $f(x) = \sqrt{x}$, (c) $f(x) = \dfrac{1-x}{2+x}$

- Other Notations $\quad y = f(x)$

derivative $\quad f'(x) = y' = \dfrac{dy}{dx} = \dfrac{df}{dx} = \dfrac{d}{dx} f = Df(x) = D_x f(x)$

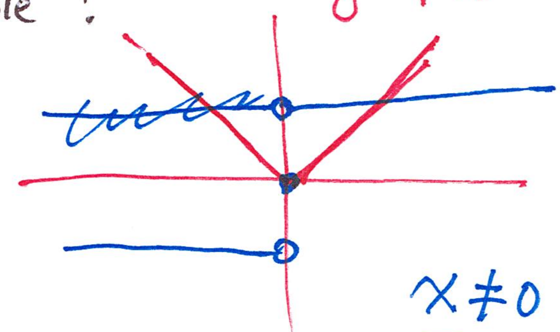
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Def. A function f is differentiable at a $\iff$ $f'(a)$ exists.

f is differentiable on an open interval (a,b) $\iff$ $f'(x)$ exists for all $x \in (a,b)$

Ex. 5 Where is $f(x) = |x|$ differentiable?

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$



$x \neq 0$

Theorem f is differentiable at $a \implies f$ is continuous at a.

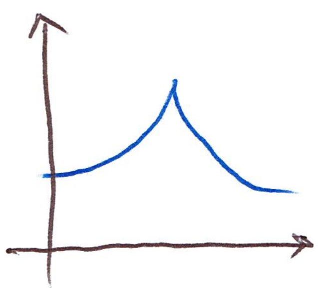
$$f'(a) = \lim_{\substack{\cancel{h\to 0} \\ x\to a}} \frac{f(\overset{x}{\cancel{a+h}}) - f(a)}{x-a} \qquad \lim_{\substack{x \to a}} \left[f(x) - f(a)\right] = 0$$

Proof

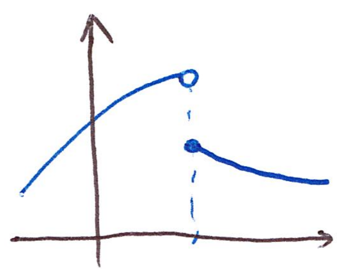
$$0 \overset{?}{=} \lim_{x\to a} \left(f(x) - f(a)\right) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot (x-a)$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a} (x-a)$$

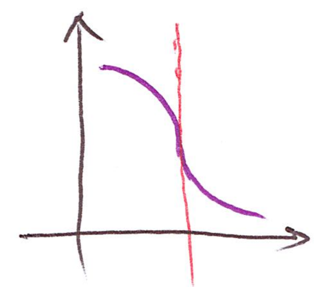
- **Not Differentiable Functions**



corner



discontinuity



vertical tangent

- Higher Derivatives

the 2^{nd} derivatives $\quad f''(x) = \left(f'(x)\right)' = \dfrac{d^2 f}{dx^2}$

the 3^{rd} derivatives $\quad f'''(x) = \left(f''(x)\right)' = \dfrac{d^3 f}{dx^3}$

Ex. 6&7 $f(x) = x^3 - x$, $\quad f''(x) = ?$, $f'''(x) = ?$, $f^{(4)}(x) = ?$

Chapter 3 Differentiation Rules (10 lectures)

§3.1 Derivatives of Polynomials and Exponential Functions

- $\boxed{\frac{d}{dx}(c) = 0,}$ c - constant $\quad \frac{d}{dx} c = \lim_{h \to 0} \frac{c - c}{h} = 0$

- $\boxed{\frac{d}{dx}(x^n) = n x^{n-1},}$ (1) $n > 0$ integer ; (2) n is a real number

$a^2 - b^2 = (a+b)(a-b)$

$\underline{n=1}$ $\quad \frac{d}{dx}(x) = \lim_{h \to 0} \frac{(\cancel{x}+h) - \cancel{x}}{h} = 1$

$\underline{n=2}$ $\quad \frac{d}{dx}(x^2) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{\cancel{h}} = \frac{(2x+h)\cancel{h}}{} = 2x$

$\underline{n=3}$ $\quad \frac{d}{dx}(x^3) =$

$\underline{n=-1}$ $\quad \frac{d}{dx}\left(x^{-1}\right) =$

$\underline{n=\frac{1}{2}}$ $\quad \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

<u>Examples</u>

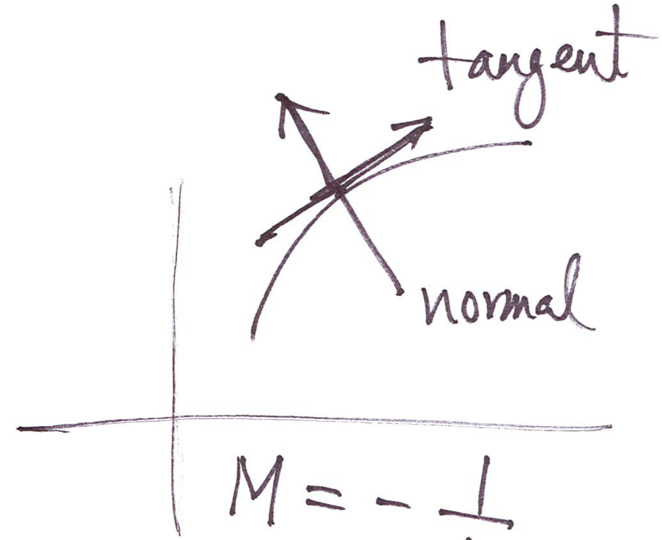
$f(x) = \frac{1}{x^2}$, $\quad f'(x) = \left(x^{-2}\right)' = -2\, x^{-2-1}$

$y = \sqrt[3]{x^2}$, $\quad \frac{dy}{dx} = \left(x^{\frac{2}{3}}\right)' = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} x^{-\frac{1}{3}}$

Ex. 3 Find equations of the tangent and normal line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$. Illustrate by graphing the curve and these lines.

$$y = x^{\frac{3}{2}}, \quad y' = \frac{3}{2} x^{\frac{3}{2}-1} = \frac{3}{2} x^{\frac{1}{2}}, \quad y'(1) = \frac{3}{2} = m$$

$$y - 1 = \frac{3}{2}(x - 1)$$



$$M = -\frac{1}{m}$$

$$y - 1 = -\frac{2}{3}(x - 1)$$

Ex. 6 Find the points on the curve $y = x^4 - 6x^2 + 4$, where the tangent line is horizontal.

$$y'(x) = 0 = 4x^3 - 12x = 4x\left(x^2 - 3\right) = 4x\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right)$$

$$x = 0,\ -\sqrt{3},\ \sqrt{3}$$

$$y = 4,\ -5,\ -5$$

Ex. 7 The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, s – centimeters, t – seconds

(a) Find the acceleration as a function of time. (b) What is the acceleration after 2 seconds?

$$a(t) = s''(t) = \left(6t^2 - 10t + 3\right)' = 12t - 10$$

$$a(2) = 24 - 10 = 14$$

- $\frac{d}{dx}\left(c\,f(x)\right) = c\,\frac{d}{dx}f(x)$

- $\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

<u>example</u>

$$\frac{d}{dx}\left(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5\right)$$

$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

- exponential functions $\quad f(x) = a^x$, $\quad$ base $a > 0$

$$\boxed{f'(x) = f'(0)\, a^x}$$

$$\| $$

$$\lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x\,[a^h - 1]}{h} = a^x \boxed{\lim_{h\to 0} \frac{a^h - 1}{h}}$$

$\underline{a=2}$ $\quad f'(0) = \lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.69 \Longrightarrow \frac{d}{dx} 2^x \approx 0.69\ 2^x$

$\underline{a=3}$ $\quad f'(0) = \lim_{h\to 0} \frac{3^h - 1}{h} \approx 1.10 \Longrightarrow \frac{d}{dx} 3^x \approx 1.10\ 3^x$

- Number e

e is the number such that $\underline{\lim_{h\to 0} \frac{e^h - 1}{h} = 1}$

$$\boxed{\frac{d}{dx} e^x = e^x}$$

$$\frac{d}{dx}(a^x) = \square\, a^x$$

Ex. 8 $f(x) = e^x - x$, compute f' and f''.

$$f' = e^x - 1$$

$$f'' = e^x$$

Ex. 9 At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$? $\rightarrow m = 2$

$$y' = (e^x)' = e^x = 2 \implies x = \ln 2$$

$$y = e^{\ln 2} = 2$$

$(2, 2)$

$$\boxed{e^x = 2}$$