$$\frac{d}{dx} x^r = r x^{r-1}$$

$$\boxed{\frac{d}{dx} e^x = e^x}$$

$$a^x = e^{\ln a^x} = e^{x \ln a}$$

$$\frac{d}{dx} a^x \stackrel{?}{=} a^x \ln a$$

$$\parallel$$

$$\frac{d}{dx} e^{x \ln a} = e^{x \ln a} (x \ln a)' = e^{x \ln a} \ln a$$

$$= a^x \ln a$$

§3.2 The Product and Quotient Rules

product rule

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x+h) + f(x)\frac{g(x+h)-g(x)}{h}\right]$$

Examples

(1) $f(x) = xe^x$, compute $f'(x)$ and $f^{(n)}(x)$

$$f'(x) = e^x + xe^x = (1+x)e^x$$

$$f''(x) = e^x + (1+x)e^x = (2+x)e^x$$

$$f'''(x) = e^x + (2+x)e^x = (3+x)e^x$$

$$f^{(n)}(x) = (n+x)e^x$$

(2) $f(t) = \sqrt{t}\,(a + bt)$, compute $f'(t)$

$$= a t^{\frac{1}{2}} + b t^{\frac{3}{2}}$$

$$f'(t) = a \cdot \frac{1}{2} t^{\frac{1}{2}-1} + b \cdot \frac{3}{2} t^{\frac{3}{2}-1}$$

$$= \frac{a}{2} t^{-\frac{1}{2}} + \frac{3b}{2} t^{\frac{1}{2}} = \frac{a}{2\sqrt{t}} + \frac{3b}{2}\sqrt{t}$$

(3) $f(x) = \sqrt{x}\, g(x)$, $g(4) = 2$, $g'(4) = 3$. Compute $f'(4)$

$$= x^{\frac{1}{2}} g(x)$$

$$f'(x)\Big|_{x=4} = \left[\frac{1}{2} x^{-\frac{1}{2}} g(x) + x^{\frac{1}{2}} g'(x)\right]_{x=4}$$

$$= \frac{1}{2} \cdot 4^{-\frac{1}{2}} g(4) + 4^{\frac{1}{2}} g'(4)$$

$$= \frac{1}{2 \cdot 2} \cdot 2 + 2 \cdot 3 = \frac{1}{2} + 6$$

quotient rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{\left(g(x)\right)^2}$$

$$\parallel$$

$$\lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{g(x)f(x+h) - g(x)f(x) - f(x)g(x+h) + g(x)f(x)}{h\,g(x+h)\,g(x)}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h\,g(x+h)} - f(x)\frac{g(x+h)-g(x)}{h\,g(x+h)\,g(x)}\right]$$

Ex. 4 $y = \frac{x^2+x-2}{x^3+6}$, compute y'.

$$y' = \frac{(2x+1)(x^3+6) - (x^2+x-2)\cdot 3x^2}{(x^3+6)^2}$$

Ex. 5 $f(x) = \frac{3x^2+2\sqrt{x}}{x}$, compute $f'(x) = 3 + 2\cdot\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$

$= 3x + 2x^{-\frac{1}{2}}$ $\qquad = 3 - x^{-\frac{3}{2}}$

§3.3 Derivatives of Trigonometric Functions

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

$$\| $$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cosh + \underline{\sin h \cos x} - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \underline{\cos x} + \lim_{h \to 0} \frac{\cos h - 1}{h} \sin x$$

Ex. 1 $y = x^2 \sin x$, compute $y' = 2x \sin x + x^2 \cos x$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

• $(\sin x)' = \cos x$, • $(\cos x)' = -\sin x$, • $(\tan x)' = \sec^2 x$,

• $(\cot x)' = -\csc^2 x$, • $(\sec x)' = \sec x \tan x$, • $(\csc x)' = -\csc x \cot x$.

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x - (-\sin x)\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{\sin^2 x}{\cos^2 x} - \frac{1}{\cos^2 x} = \frac{\sin^2 x - 1}{\cos^2 x} = -1$$

<u>Ex. 2</u> $f(x) = \dfrac{\sec x}{1+\tan x}$, $f'(x) = ?$ For what values of x does the graph of f have a horizontal tangent?

$$f'(x) = \frac{\sec x \tan x\,(1+\tan x) - \sec x \cdot \sec^2 x}{(1+\tan x)^2} = \frac{\sec x\left[\tan x + \underline{\tan^2 x - \sec^2 x}\right]}{(1+\tan x)^2}$$

$$= \frac{\sec x\,(\tan x - 1)}{(1+\tan x)^2}$$

$$f'(x) = 0 \Rightarrow \frac{\sec x\,(\tan x - 1)}{(1+\tan x)^2} = 0$$

$\sec x = 0$ or $\underline{\tan x = 1}$

Ex. 4 $f(x) = \cos x$, compute $f^{(27)}(x) = \sin x$

$f' = -\sin x$

$f'' = -\cos x$

$f''' = \sin x$

$f^{(4)} = \cos x$

Ex. 5 $\lim_{x \to 0} \frac{\sin 7x}{4x} = \left(\lim_{x \to 0} \frac{\sin 7x}{7x} \right) \cdot \frac{7}{4} = \frac{7}{4}$

$0 \cdot \infty$

Ex. 6 $\lim_{x \to 0} x \cot x = \lim_{x \to 0} \frac{x \cos x}{\sin x} = \lim_{x \to 0} \frac{x}{\sin x} \lim_{x \to 0} \cos x = 1$