Study Guide for Exam 2

1.1 Compute derivatives

(i) $y = \sin(\sin(\sin x))$, $y' = \cos(\sin(\sin x))\cos(\sin x)\cos x$

(ii) $y = \left(\frac{t-2}{2t+1}\right)^9$, $y' = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{1\cdot(2t+1)-(t-2)\cdot 2}{(2t+1)^2}$

(iii) $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$, $y' = \frac{d}{dx}\left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{\frac{1}{2}} = \frac{1}{2}\left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}} \cdot \left[1+\frac{1}{2}(x+x^{\frac{1}{2}})^{-\frac{1}{2}} \cdot (1+\frac{1}{2}x^{-\frac{1}{2}})\right]$

$= \frac{1}{2}\frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \left[1+\frac{1}{2}\frac{1+\frac{1}{2}\frac{1}{\sqrt{x}}}{\sqrt{x+\sqrt{x}}}\right]$

(iv) $y = e^{\sec(3\theta)}$, $y' = e^{\sec(3\theta)} \cdot \sec(3\theta)\tan(3\theta)\cdot 3$

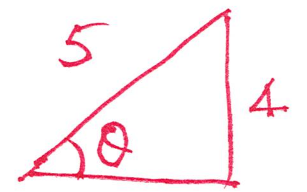
$\left[a^{f(x)}\right]' = a^{f(x)} \cdot \ln a \cdot f'(x)$

(v) $y = e^{2^{x^3}}$, $y' = e^{2^{x^3}} \cdot \left(2^{x^3}\cdot \ln 2\right)\cdot 3x^2$

2.1 Find the exact values

$\theta = \sin^{-1}\left(\frac{4}{5}\right) \Longleftrightarrow \sin\theta = \frac{4}{5}$

(i) $\tan\left(\sin^{-1}\left(\frac{4}{5}\right)\right) = \tan\theta = \frac{4}{3}$



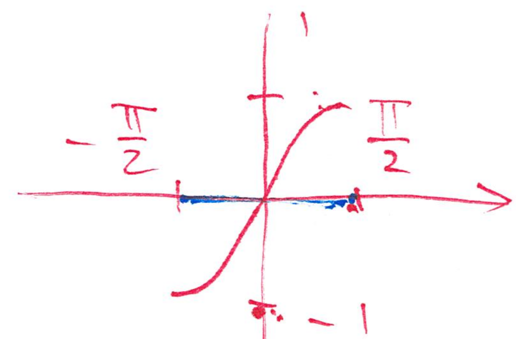
$\sqrt{25-16} = 3$

$\sin\left(\frac{7\pi}{3}\right) = \sin\frac{6\pi+\pi}{3}$

(ii) $\sin^{-1}\left(\sin\left(\frac{7\pi}{3}\right)\right) \neq \frac{7\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $= \sin\left(2\pi + \frac{\pi}{3}\right)$

$= \sin\left(\frac{\pi}{3}\right)$

$= \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$

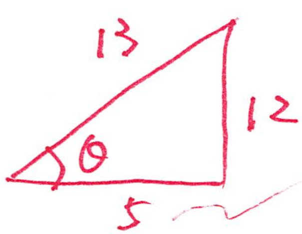


$\mathrm{dom}(\sin^{-1}x) = [-1, 1]$

$\mathrm{range}(\sin^{-1}x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) $\sin\left(2\sin^{-1}\left(\frac{12}{13}\right)\right) = \sin(2\theta) = 2\sin\theta\cos\theta$

$\theta = \sin^{-1}\left(\frac{12}{13}\right) \Longleftrightarrow \sin\theta = \frac{12}{13}$ $= 2\cdot\frac{12}{13}\cdot\frac{5}{13}$



$\sqrt{13^2-12^2} = \sqrt{25\cdot 1} = 5$

1.2 $F(x) = f(x)^2 \cdot f(g(x))$ and $\begin{cases} f(1)=5, & f(2)=3, & f(3)=-1 \\ f'(1)=4, & f'(2)=3, & f'(3)=-2 \\ g(1)=3, & g(2)=2, & g(3)=-1 \\ g'(1)=2, & g'(2)=3, & g'(3)=4 \end{cases}$

Find $F'(1)$.

$$F'(x) = 2f(x)\cdot f'(x)\cdot f(g(x))$$
$$+ f(x)^2 \cdot f'(g(x)) \cdot g'(x)$$

$$F'(1) = 2f(1)f'(1)\,f(g(1)) + f(1)^2 \cdot f'(g(1)) \cdot g'(1)$$

$$= 2\cdot 5\cdot 4\cdot f(3) + 5^2 \cdot f'(3)\cdot 2$$

$$= 40\cdot(-1) + \cancel{25}\ 50\cdot(-2) = -140$$

3.1 Find the derivatives $\quad y = f(x)^{g(x)}$

(i) $y = x^x$, $y' = \left(e^{\ln x^x}\right)' = \left(e^{x\ln x}\right)'$

$= e^{x\ln x} \cdot \left(\ln x + x \cdot \frac{1}{x}\right)$

(ii) $y = (\ln x)^{\tan(3x)} = e^{\ln(\ln x)^{\tan(3x)}} = e^{\tan(3x)\cdot\ln(\ln x)}$

$y' = (\ln x)^{\tan(3x)} \left[\sec^2(3x)\cdot 3 \cdot \ln(\ln x) + \tan(3x)\cdot\frac{1}{\ln x}\cdot\frac{1}{x}\right]$

(iii) $y = (\sqrt{x})^{\sin x} = e^{\ln \sqrt{x}^{\sin x}} = e^{\sin x \cdot \cancel{\ln} \frac{1}{2}\ln x}$

$y' = (\sqrt{x})^{\sin x} \frac{1}{2}\left[\cos x \cdot \ln x + \sin x \cdot \frac{1}{x}\right]$

4.1 $\left[f(x) + x^2 f(x)^3 = 10\right]'$ and $f(1) = 2$. Find $f'(1)$.

$$f'(x) + 2x \cdot f(x)^3 + \underbrace{x^2\, 3 \cdot (f(x))^2 \cdot f'(x)} = 0$$

$$f'(x)\Big|_{x=1} = \frac{-2x\, f(x)^3}{1 + 3x^2 f(x)^2}\Big|_{x=1} = \frac{-2 \cdot 1 \cdot f(1)^3}{1 + 3 \cdot 1^2 \cdot f(1)^2} = \frac{-16}{13}.$$

4.2 Find the slope of the tangent to the curve given by $\left(x^2 + 2xy(x) - y(x)^2 + x = 2\right)'$ at $(x,y) = (1,2)$

$$2x + 2(y + xy') - 2y(x)\,y' + 1 = 0$$

$$(2x - 2y)\, y' = -2x - 2y - 1$$

$$y'\Big|_{(x,y)=(1,2)} = \frac{2x + 2y + 1}{2y - 2x}\Big|_{(x,y)=(1,2)}$$

4.3 Find $\frac{dy}{dx}$ given $\left(e^{\frac{x}{y(x)}} = 7x - y(x)\right)'$

$$e^{\frac{x}{y}} \cdot \frac{1 \cdot y - x \cdot y'}{y^2} = 7 - y'$$

$$\| $$

$$e^{\frac{x}{y}} \cdot y^{-2}(y - xy')$$

$$\left(1 - xy^{-2} e^{\frac{x}{y}}\right) y' = -y^{-1} e^{\frac{x}{y}}$$

$$y' = \frac{y^{-1} e^{\frac{x}{y}}}{xy^{-2} e^{\frac{x}{y}} - 1}$$

5.1 Find the linear approx $L(x)$ of the function $f(x) = e^x$ at $a=0$. Use this to estimate $e^{0.01}$.

$L(x) = f(a) + f'(a)(x-a)$ — linear approx. of $f(x)$ at a

$L(x) = e^0 + e^0 \cdot (x-0) = 1 + x$

$e^{0.01} \approx L(0) = 1 + 0.01 = \underline{1.01}$

$\sqrt[3]{27}$

5.2 Estimate the value of $\sqrt[3]{26.8}$ using a linear approximation.

$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$, at $\underline{a=27}$

$f'(x) = \frac{1}{3} \cdot x^{-\frac{2}{3}}$

$L(x) = f(a) + f'(a)(x-a) = 27^{\frac{1}{3}} + \frac{1}{3} \cdot 27^{-\frac{2}{3}} \cdot (x-27)$

$\underline{\underline{\sqrt[3]{26.8}}} \approx L(26.8) = 3 + \frac{1}{3} \cdot \frac{1}{3^2} \cdot (26.8 - 27)$

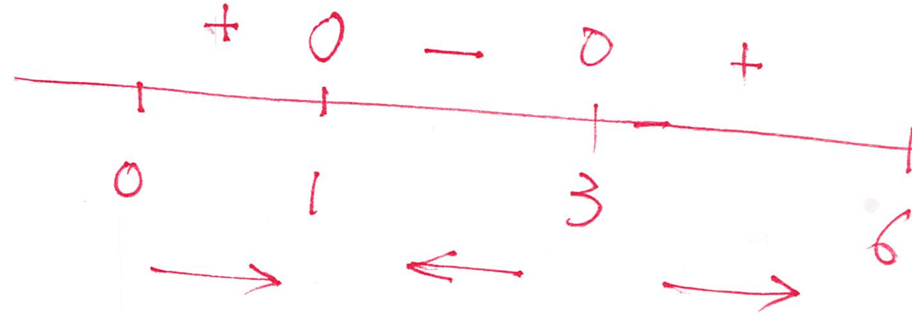
$= 3 + \frac{1}{27} \cdot (-0.2)$

6.2 The position of a particle is given by the function $s = f(t) = t^3 - 6t^2 + 9t$.

Find the total distance traveled ~~by~~ during the first ~~seconds~~ 6 seconds.

$$v(t) = f'(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3)$$

$$= 3(t-1)(t-3)$$



$$|f(1) - f(0)| = 4$$

$$|f(3) - f(1)| = 4$$

$$|f(6) - f(3)| = 54$$

$$62$$

$f(0) = 0$

$f(1) = 4$

$f(3) = 3^3 - 6 \cdot 3^2 + 9 \cdot 3$

$= -3^3 + 3^3 = 0$

$f(6) = 6^3 - 6 \cdot 6^2 + 9 \cdot 6$

$= 54$

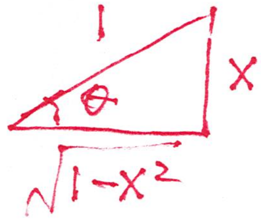
7.1 Find the formula for the following expression.

(i) $\tan\left(\sin^{-1}x\right) = \tan\theta = \dfrac{x}{\sqrt{1-x^2}}$

$\theta = \sin^{-1}x$

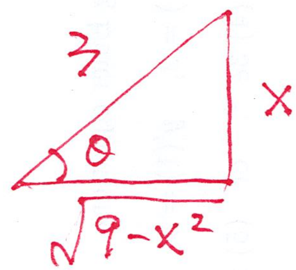
$\Updownarrow$

$\sin\theta = x$



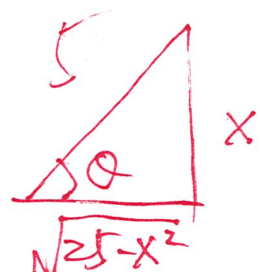
(ii) $\cos\left(\underbrace{\tan^{-1}\dfrac{x}{\sqrt{9-x^2}}}_{\theta}\right) = \cos\theta = \dfrac{\sqrt{9-x^2}}{3}$

$\tan\theta = \dfrac{x}{\sqrt{9-x^2}}$



(iii) $\csc\left(\underbrace{\cot^{-1}\dfrac{\sqrt{25-x^2}}{x}}_{\theta}\right)$ when $x>0$ $= \csc(\theta) = \dfrac{1}{\sin\theta} = \dfrac{5}{x}$

$\cot\theta = \dfrac{\sqrt{25-x^2}}{x}$



8.1 Find the exact value

$\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$, $\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$

(i) $\sinh(0) = \dfrac{e^{0} - e^{-0}}{2} = \dfrac{1-1}{2} = 0$

(ii) $\sinh(\ln 5) = \frac{1}{2}\left(e^{\ln 5} - e^{-\ln 5}\right) = \frac{1}{2}\left(5 - \frac{1}{5}\right)$

(iii) $\dfrac{1+\tanh\left(\frac{1}{2}\right)}{1-\tanh\left(\frac{1}{2}\right)} =$

8.2 $f(x) = \sinh(\underline{\ln x})$,

$f'(x) = \cosh(\ln x) \cdot \dfrac{1}{x}$

$f'(5) = \cosh(\ln 5) \cdot \dfrac{1}{5}$

#14 If a snow ball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm

$$\frac{dS}{dt} = 1 \ cm^2/min \qquad \left.\frac{d\ell}{dt}\right|_{\ell=10} = ?$$

$$S = 4\pi r^2 = \pi (2r)^2 = \pi \ell^2$$

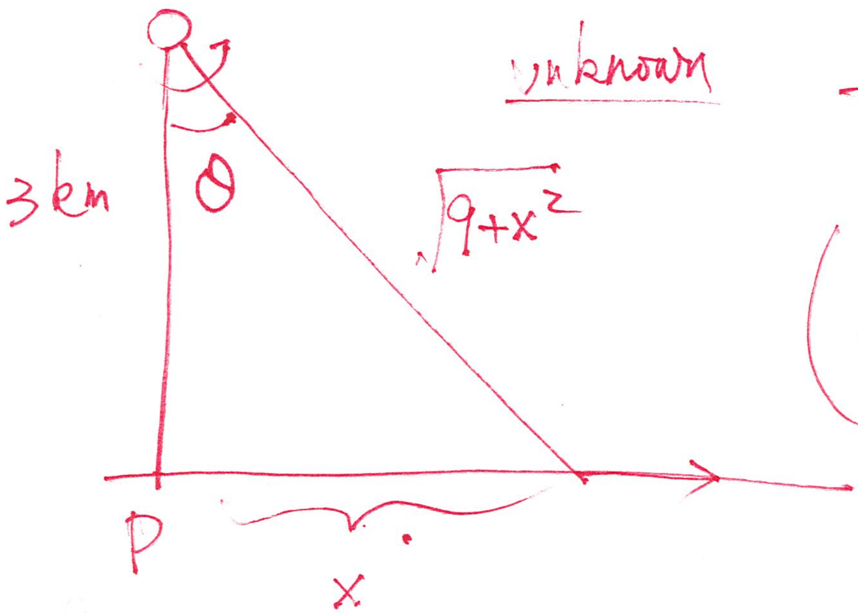
$$\frac{dS}{dt} = \pi \cdot 2\ell \frac{d\ell}{dt} \Rightarrow \left.\frac{d\ell}{dt}\right|_{\ell=10} = \left.\frac{1}{2\pi\ell}\frac{dS}{dt}\right|_{\ell=10}$$

$$= \frac{1}{20\pi} \cdot 1 = \frac{1}{20\pi} \ cm/min$$

Light House Problem (P251, #44)

A light house is located on a small island 3 km away from the nearest point P on a straight shoreline and its lights makes 4 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

Given $\frac{d\theta}{dt} = 4\cdot(2\pi)$ rad/min



unknown $\left.\frac{dx}{dt}\right|_{x=1\text{ km}} = ?$

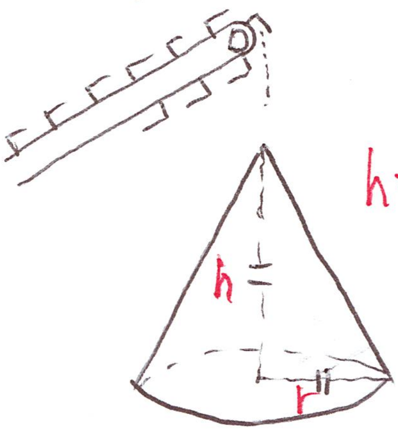
$$\left(\tan\theta_{(t)} = \frac{x(t)}{3} \right)'$$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{3}\frac{dx}{dt}$$

$$\left.\frac{dx}{dt}\right|_{x=1} = 3\sec^2\theta \cdot \frac{d\theta}{dt} = 3(1+\tan^2\theta)\frac{d\theta}{dt} = 3\left(1+\left(\frac{x}{3}\right)^2\right)\frac{d\theta}{dt}$$

Gravel Problem (P250, #29)

Gravel is being dumped from a conveyor belt at a rate of 30 ft³/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the ~~pile~~ pile is 10 ft high?



$h = 2r$

$$\frac{dV}{dt} = 30\ ft^3/min, \qquad \left.\frac{dh}{dt}\right|_{h=10} = ?$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$$

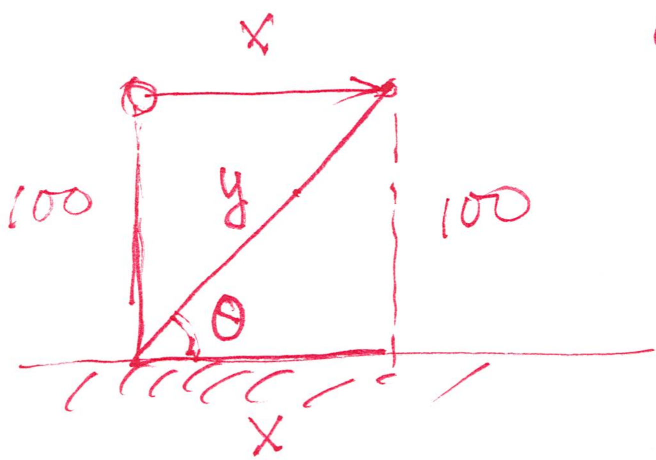
$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$\left.\frac{dh}{dt}\right|_{h=10} = \left.\frac{4}{\pi h^2}\frac{dV}{dt}\right|_{h=10} =$$

Kite Problem (P251, #30)

A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and horizontal decreasing when 200 ft of string has been let out?

$$\left.\frac{d\theta}{dt}\right|_{y=200} = ? \qquad \frac{dx}{dt} = 8\ ft/s$$



$$\left(\tan\theta = \frac{100}{x}\right)' \qquad (x^{-1})' = -1\,x^{-2}$$

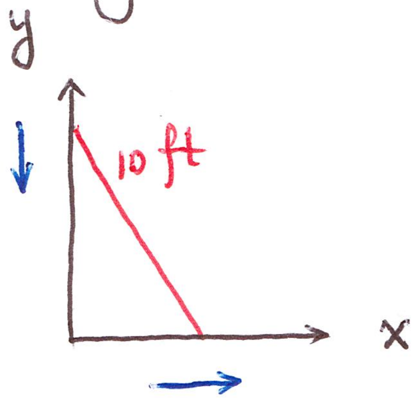
$$\sec^2\theta\,\frac{d\theta}{dt} = -\frac{100}{x^2}\cdot\frac{dx}{dt}$$

$$\left.\frac{d\theta}{dt}\right|_{y=200} = -\frac{100}{x^2\sec^2\theta}\left.\frac{dx}{dt}\right|_{y=200} = -\frac{100}{\qquad}$$

$$y = 200 \Rightarrow x = \sqrt{200^2 - 100^2} = 100\sqrt{3}$$

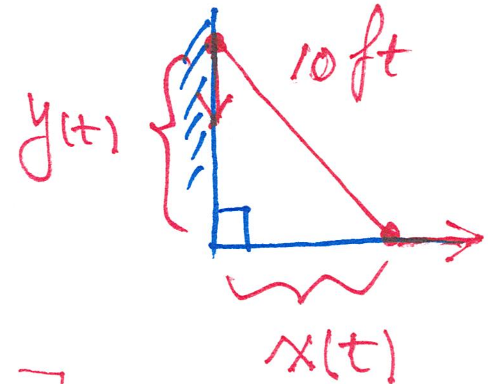
$$\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{100}{100\sqrt{3}}\right)^2 = 1 + \frac{1}{3}$$

<u>Ex. 2</u> A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladd[er] sliding down the wall when the bottom of the ladder is 6 ft from the wall?



Given $\frac{dx}{dt} = 1$ ft/s

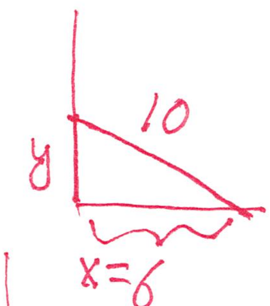
Unknown $\left.\frac{dy}{dt}\right|_{x=6} = ?$



relation

$$\frac{d}{dt}\left[x^2(t) + y^2(t) = 10^2\right] \quad \text{relation}$$

$$2x(t)\cdot\frac{dx}{dt} + 2y(t)\frac{dy}{dt} = 0$$

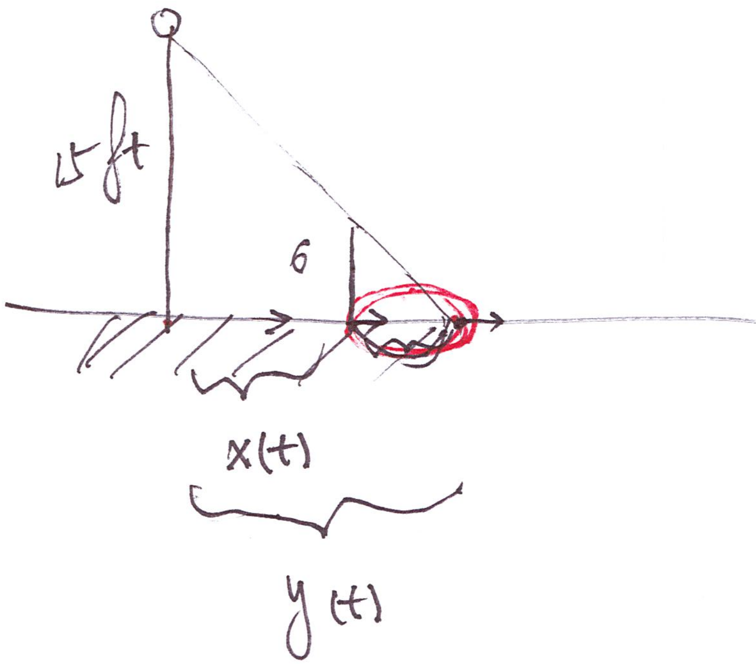


$$\left.\frac{dy}{dt}\right|_{x=6} = \frac{-2x(t)\frac{dx}{dt}}{2y(t)} = -\left.\frac{x(t)}{y(t)}\frac{dx}{dt}\right|_{x=6}$$

$$= -\frac{6}{\sqrt{10^2-6^2}}\cdot 1 = -\frac{6}{\sqrt{16\cdot 4}} = -\frac{6}{8} = -\frac{3}{4}$$

#15 A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

$$\frac{dx}{dt} = 5 \text{ ft/s}, \quad \left.\frac{dy}{dt}\right|_{x=40} = ?$$



$$\frac{6}{15} = \frac{y-x}{y}$$

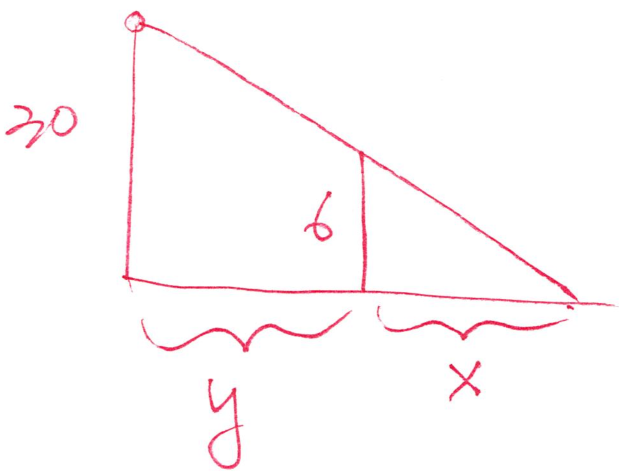
$$\left(1 - \frac{6}{15}\right) y = x$$

$$y = \frac{5}{3} x$$

$$\frac{dy}{dt} = \frac{5}{3} \frac{dx}{dt} = \frac{5}{3} \cdot 5 = \frac{25}{3}$$

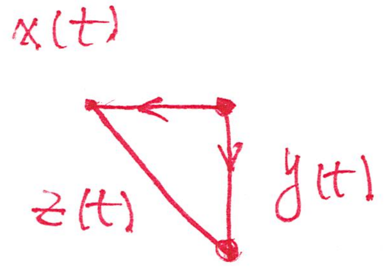
(Quiz 7, #1)

Bill walks away from a street light whose lamp is 30 feet above ground. If Bill is 6 feet tall and his shadow is lengthening at the rate of 2 feet per second, how fast is he walking?



$$\frac{dx}{dt} = 2 \text{ ft/s}. \qquad \left.\frac{dy}{dt}\right| = ?$$

#17 Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?



$$\frac{dy}{dt} = 60 \text{ mi/h}, \quad \frac{dx}{dt} = 25 \text{ mi/h}$$

$$\left.\frac{dz}{dt}\right|_{t=2} = ?$$

$$z^2 = x^2 + y^2$$

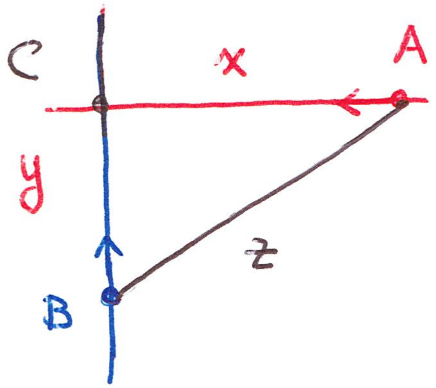
$$2z \cdot z' = 2x \cdot x' + 2y \cdot y' \Rightarrow \left. z' \right|_{t=2} = \left. \frac{x x' + y y'}{z} \right|_{t=2}$$

$x(2) = 2 \cdot 25 = 50$ mi

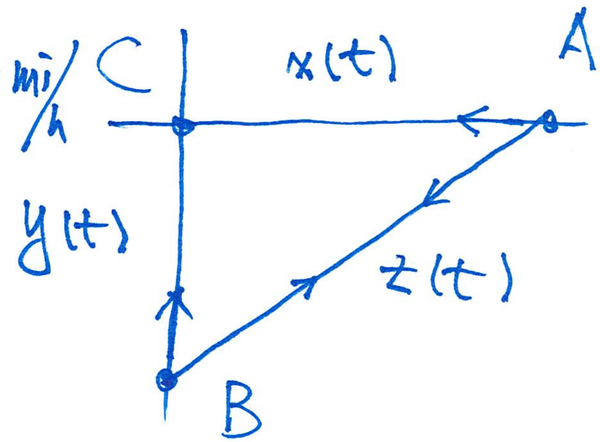
$y(2) = 2 \cdot 60 = 120$ mi

$z(2) = \sqrt{x(2)^2 + y(2)^2} = \sqrt{}$

Ex. 4 Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?



Given $\frac{dx}{dt} = -50 \text{ mi/h}$, $\frac{dy}{dt} = -60 \text{ mi/h}$



Unknown $\left.\frac{dz}{dt}\right|_{(0.3=x,\ y=0.4)} = ?$

relation

$$\frac{d}{dt}\left[z^2(t) = x^2(t) + y^2(t)\right] \quad \text{relation}$$

$$2z \cdot \frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\left.\frac{dz}{dt}\right|_{(x=0.3,\ y=0.4)} = \left.\frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)\right|_{(x=0.3,\ y=0.4)}$$

$$= \frac{1}{\sqrt{(0.3)^2+(0.4)^2}}\left[0.3\cdot(-50) + 0.4(-60)\right]$$