Study Guide for Exam 3

1. Find the absolute max/min and local max/min of

(1) $f(x) = 3x^4 - 16x^3 + 18x^2$ on $[-1, 4]$

critical numbers

$f'(x) = 12x^3 - 3\cdot 16x^2 + 2\cdot 18x$

$= 12x(x^2 - 4x + 3x) = 12x(x-1)(x-3) = 0$

$f'(x)$: $-$ 0 $+$ 0 $-$ 0 $+$

-1 $\quad$ 0 $\quad$ 1 $\quad$ 3 $\quad$ 4

l. min $\quad$ l. max $\quad$ l. min

$f(0) = \underline{0}$, $f(1) = 3 - 16 + 18 = \underline{5}$, $f(3) = \underline{?}$

$[-1, 4]$ $\qquad f(-1) = 3 + 16 + 18 = \underline{37}$

$f(4) = \underline{?}$

~~1/24~~ (2) $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1) = 0$

$+ \quad - \quad +$

$-2 \quad -1 \quad 2 \quad 3$

l. max $\quad$ l. min

$f(-1) = -2 - 3 + 12 + 1 = 8$ l. max $\Rightarrow$ abs. max.

$f(2) = 16 - 12 - 24 + 1 = -19$ l. min $\Rightarrow$ abs. min

$f(-2) = -16 - 12 + 24 + 1 = -3$

$f(3) = 27 - 36 + 1 = -8$

(3) $f(x) = x e^{\frac{x}{2}}$ on $[-3, 1]$

$f'(x) = e^{\frac{x}{2}} + x \cdot e^{\frac{x}{2}} \cdot \frac{1}{2} = \underline{e^{\frac{x}{2}}} \Big(\underbrace{1 + \frac{x}{2}}_{= 0} \Big) = 0$

$\Rightarrow 1 + \frac{x}{2} = 0 \Rightarrow x = -2$

$- \quad 0 \quad +$

$-3 \quad -2 \quad 1$

l. min

$f(-3) = -3 e^{-\frac{3}{2}} = -\dfrac{3}{e^{3/2}}$ abs. min

$f(-2) = -2 e^{-1} = -\dfrac{2}{e} > f(-3)$

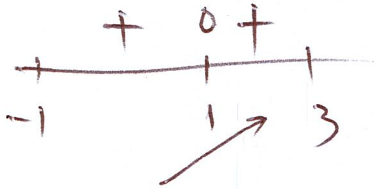
$f(1) = e^{\frac{1}{2}}$ — abs. max.

$\dfrac{3}{e} \cdot \dfrac{1}{e^{1/2}} \qquad \dfrac{2}{e}$

$\dfrac{3}{2} \qquad e^{\frac{1}{2}}$

(4) $f(x) = (x-1)^3$ on $[-1, 3]$

$f'(x) = 3(x-1)^2 = 0 \Rightarrow x = 1$



$f(-1) = -8$ abs. min.

$f(1) = 0$

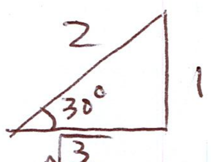
$f(3) = 8$ abs. max

(5) $f(x) = 2\cos x + \sin(2x)$ on $[0, \frac{\pi}{2}]$

$f' = -2\sin x + 2\cos(2x) = 2\left[-\sin x + \cos^2 x - \sin^2 x\right] = -2\left[2\sin^2 x + \sin x - 1\right] = 0$

$\sin x = \dfrac{-1 \pm \sqrt{1^2 - 4\cdot 2\cdot(-1)}}{2\cdot 2} = \dfrac{-1 \pm 3}{4} = -1$ or $\frac{1}{2}$

$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$



$f(0) = 2$, $f(\frac{\pi}{6}) = 2\cdot\frac{\sqrt{3}}{2} + \sin\frac{\pi}{3} = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$ abs. max

$f(\frac{\pi}{2}) = 0 + \sin(\pi) = 0$ abs. min

(6) $f(x) = \ln(x^2 + x + 1)$ on $[-1, 1]$

$f'(x) = \dfrac{2x+1}{x^2+x+1} = 0 \Rightarrow x = -\frac{1}{2}$

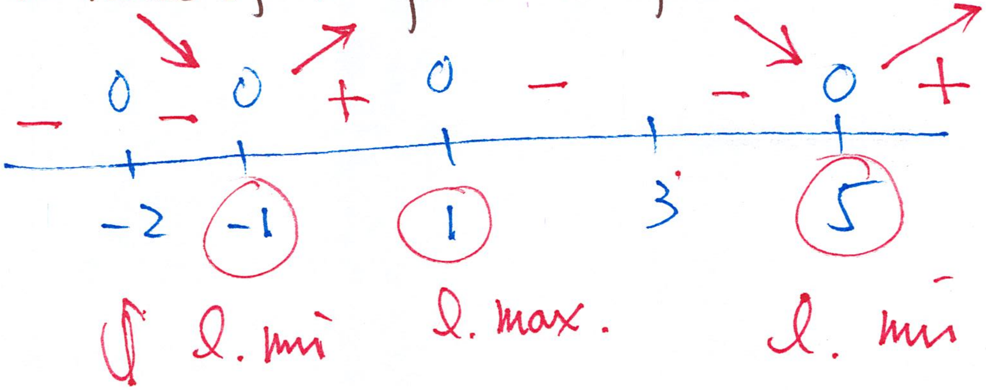
$f(-1) = \ln 1 = 0$

$f(-\frac{1}{2}) = \ln\left(\frac{1}{4} - \frac{1}{2} + 1\right) = \ln\frac{3}{4} = \ln 3 - \ln 4$ abs. min

$f(1) = \ln 3$ abs. max

2.1 $f'(x) = (x+2)^2(x+1)(x-1)^3(x-3)^2(x-5) = 0$

Find the values of x for which f takes local max/min.



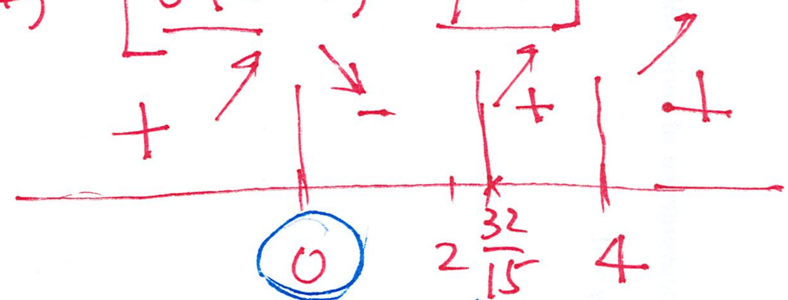
2.2 (a) Find the critical numbers of $f(x) = x^8(x-4)^7$

(b) What does the 2^{nd}-Der. test tell you about the behavior of f at these critical numbers?

(c) What does the 1^{st}-Der. test tell you that the 2^{nd}-Der. test does not?

$$f'(x) = 8x^7(x-4)^7 + 7x^8(x-4)^6 = x^7(x-4)^6\left[8(x-4)+7x\right]$$

$$= x^7(x-4)^6(15x-32) = 0$$



$x = 0, \frac{32}{15}, 4$

l. max l. min

$$f''(x) = 7x^6(x-4)^6(15x-32) + 6x^7(x-4)^5(15x-32) + 15x^7(x-4)^6.$$

$f''(0) = 0$, $f''(4) = 0$

$$f''\left(\frac{32}{15}\right) = 15\cdot(\ )^7(x-4)^6 > 0 \Rightarrow \text{l. min}$$

3. $f''(x) = (x+7)^6 (x+1)^5 x^3 (x-4)^2$

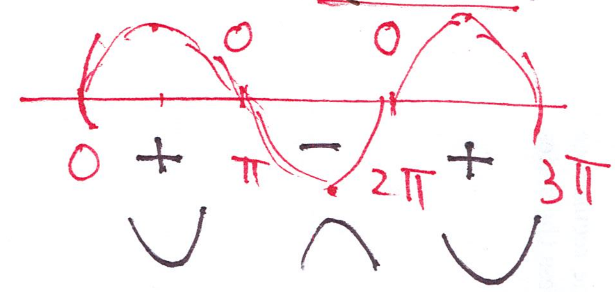
how many points of inflection does the graph of $y = f(x)$ have on $(-\infty, \infty)$?

$+ \quad 0 \quad + \quad 0 \quad - \quad 0 \quad + \quad 0 \quad +$

$-7 \qquad -1 \qquad 0 \qquad 4$

3.1 Determine how the concavity changes for $f(x) = \frac{1}{2}x - \sin x$ on $(0, 3\pi)$.

$f'(x) = \frac{1}{2} - \cos x$; $f''(x) = \sin x$



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$y = x^2$

$y'' = 2 > 0$

3.2 $f''(x) = (x+5)^3 (x+2)^2 (x-2)^5 (x-3)^3 (x-6)^2$

find the x-coordinates of all the inflection points.

− 0 + 0 + 0 − 0 + 0 +

−5 −2 2 3 6

4. (1) $$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim \frac{2\sqrt{x}}{x} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0$$

(2) $$\lim_{x\to 0}\frac{1-\cos x}{3x^2} = \lim \frac{\sin x}{6x} = \frac{1}{6} = \lim_{x\to 0}\frac{\cos x}{6} = \frac{1}{6}$$

(3) $$\lim_{x\to 0}\frac{\sin x}{1-x^2} = \cancel{\lim_{x\to 0}\frac{\cos x}{-2x}} \ \frac{0}{1} = 0$$

(4) $$\lim_{x\to 0}\frac{7^x-6^x}{3^x-2^x} = \lim_{x\to 0}\frac{7^x\ln 7-6^x\ln 6}{3^x\ln 3-2^x\ln 2} = \frac{\ln 7-\ln 6}{\ln 3-\ln 2}$$

5. (1) $\lim_{x\to 0^+} \underset{\downarrow\, 0}{\sin x}\ \underset{\downarrow\, -\infty}{\ln(2x)} \overset{?}{=} 0 = \lim_{x\to 0} \frac{\ln(2x)\to\infty}{\frac{1}{\sin x}\searrow\infty} = \lim_{x\to 0} \frac{\frac{\cancel{2}}{\cancel{2}x}}{-\frac{\cos x}{\sin^2 x}}$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$= \lim_{x\to 0} -\left(\frac{\sin x}{x}\right)\frac{\sin x}{\cos x} = 0 - \underset{\overset{\parallel}{1}}{\lim \frac{\sin x}{x}}\ \underset{\overset{\parallel}{0}}{\lim \frac{\sin x}{\cos x}}$

(2) $\lim_{x\to\infty} 2x\tan\left(\frac{1}{3x}\right)$ $\quad \infty\cdot 0$

$= \lim 2x\cdot\frac{\sin\left(\frac{1}{3x}\right)}{\cos\left(\frac{1}{3x}\right)} \overset{?}{=} \lim \frac{2}{\cos\left(\frac{1}{3x}\right)}\ \lim \frac{\sin\left(\frac{1}{3x}\right)}{\frac{1}{x}}$

$= 2\cdot\lim \frac{\cos\left(\frac{1}{3x}\right)\cdot\frac{1}{3}\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} = \frac{2}{3}$

(3) $\lim_{x\to\left(\frac{\pi}{2}\right)^-} (2x-\pi)^{\nearrow 0}\tan x^{\nearrow\infty}$

$= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \sin x\ \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{2x-\pi}{\cos x} = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{2}{-\sin x} = -2$

(4) $\lim_{x\to 1}\left(\frac{x}{x-1}^{\nearrow\infty} - \frac{1}{\ln x}^{\nearrow\infty}\right) = \lim_{x\to 1}\frac{x\ln x-(x-1)\nearrow 0}{(x-1)\ln x\to 0} = \lim_{x\to 1}\frac{\ln x+\frac{x}{x}-1}{\ln x+\frac{x-1}{x}} = \lim_{x\to 1}\frac{\ln x\nearrow 0}{\ln x+1-\frac{1}{x}\searrow 0}$

$= \lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} = \lim_{x\to 1}\frac{x}{x+1} = \frac{1}{2}$

(5) $\lim_{x\to 4}\left(\frac{1}{\sqrt{x}-2}^{\nearrow\infty} - \frac{4}{x-4}^{\nearrow\infty}\right) = \lim_{x\to 4}\frac{(\sqrt{x}+2)-4}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x\to 4}\frac{1}{\sqrt{x}+2} = \frac{1}{4}$

$x-4 = (\sqrt{x}+2)(\sqrt{x}-2)$

6. (1) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{7x} = e^{\lim_{x\to\infty} 7x \ln\left(1+\frac{3}{x}\right)}$

$\lim 7\cdot\dfrac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} = 7\lim\dfrac{\frac{1}{1+\frac{3}{x}}\cdot\cancel{\frac{3}{-x^2}}}{\cancel{\frac{1}{-x^2}}}$

$1^{\infty}, \quad 0^{0}, \quad \infty^{0}$

$= e^{7\cancel{\lim}\cdot 3} = e^{21}$

(2) $\lim_{x\to\infty}\left(\frac{2x+1}{2x-1}\right)^{4x+1} = e^{\lim_{x\to\infty}(4x+1)\cdot\ln\left(\frac{2x+1}{2x-1}\right)}$

1^{∞}

$\lim_{x\to\infty}\dfrac{\ln\left(\frac{2x+1}{2x-1}\right)}{\frac{1}{4x+1}}$

$= e^{\lim_{x\to\infty}\dfrac{\frac{2x-1}{2x+1}\cdot\frac{2(2x-1)-2(2x+1)}{(2x-1)^2}}{-\frac{1}{(4x+1)^2}\cdot 4}} = e^{-1}$

$\cancel{\left[(4x+1)^{-1}\right]}$

$\cancel{=-(4x+1)^{-2}\cdot 4}$

(3) $\lim_{x\to\infty}\left(2x+e^{5x}\right)^{\frac{1}{x}} = e^{\lim\frac{\ln(2x+e^{5x})}{x}}$

∞^{0}

$= \lim\dfrac{1}{2x+e^{5x}}\cdot\left(2+5e^{5x}\right) = 5$

$= e^{5}$

(4) $\lim_{x\to 0^+}\tan(5x)^{\sin x} = \lim_{x\to 0}\dfrac{(\sin(5x))^{\sin x}}{(\cos(5x))^{\sin x}} = \lim_{x\to 0^+}\dfrac{1}{(\cos(5x))^{\sin x}}\nearrow 1 \quad \lim_{x\to 0^+}\left(\sin(5x)\right)^{\sin x}$

0^{0}

$= \cancel{\lim} e^{\lim_{x\to 0^+}\sin x\ln(\sin 5x)} = \lim\dfrac{\ln(\sin 5x)}{\frac{1}{\sin x}} = 0 = e^{0} = 1$

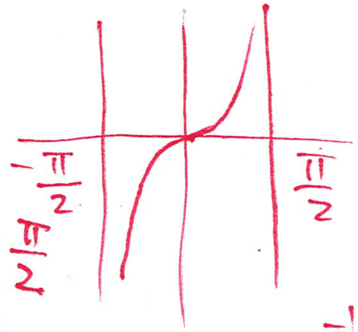
7.1 Prove that $\underbrace{\tan^{-1}x + \cot^{-1}x}_{f(x)} = \frac{\pi}{2}$ (Example 6, P291)

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \Longrightarrow f(x) = \underline{C} = f(0) = \underbrace{\tan^{-1}0}_{0} + \underbrace{\cot^{-1}0}_{\frac{\pi}{2}}$$

$\theta = \cot^{-1}0$

$\cot\theta = 0$

$\dfrac{\cos\theta}{\sin\theta}$



7.2 Determine the exact value of $\underline{\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{1}{5}\right)} = \frac{\pi}{2}$

$f(x) = \sin^{-1}x + \cos^{-1}x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0 \Longrightarrow \sin^{-1}x + \cos^{-1}x = \underline{\underline{C}} = \sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{1}{5}\right)$$

$\theta = \sin^{-1}$, $\sin\theta = 0$ $\quad$ $\theta = \cos^{-1}$, $\cos\theta = 0$ $\quad$ θ, $\sin\theta = \frac{1}{5}$

$$\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{1}{5}\right) = \sin^{-1}0 + \cos^{-1}0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

7.3 Use $f(x) = 2\sin^{-1}x - \cos^{-1}(1-2x^2)$ on ~~the~~ $[0,1]$ to determine $\boxed{2\sin^{-1}\left(\frac{1}{3}\right) - \cos^{-1}\left(1 - 2\left(\frac{1}{3}\right)^2\right)}$ $= 0$

$$f'(x) = \frac{2}{\sqrt{1-x^2}} + \frac{-4x}{\sqrt{1-(1-2x^2)^2}} = \frac{2}{\sqrt{1-x^2}} + \frac{-\cancel{4x}\,2}{\cancel{2x}\sqrt{1-x^2}} = 0$$

$a^2 - b^2 = (a+b)(a-b)$

$$1 - (1-2x^2)^2 = (2-2x^2)(2x^2) = 4x^2(1-x^2)$$

$$\Longrightarrow \underline{f(x) = C} = f(0) = 2\sin^{-1}0 - \cos^{-1}(1) = 0 - 0 = 0$$

$\cos\theta = 1$

$\Downarrow$

$\theta = 0$

9.0 Find the horizontal/vertical asymptotes

horizontal Asy.

$$\boxed{\lim_{x\to\pm\infty} f(x)} = c = y$$

vertical Asy

$$\lim_{x\to ?} f(x) = \pm\infty \qquad ? = x$$

9.1 $y = f(x) = \dfrac{2x^2}{x^2-1}$

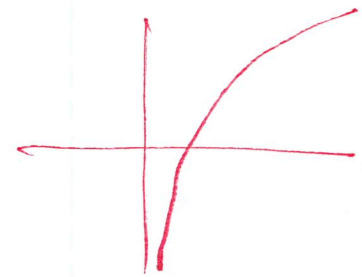
$$\lim_{x\to\infty} f(x) = 2 = y \quad \text{— H.A.}$$

$$\lim_{x\to} f(x) = \infty$$

$$\parallel$$

$$\lim_{x\to ?} \frac{2x^2}{(x+1)(x-1)}$$

$$\left.\begin{matrix} x=-1 \\ x=1 \end{matrix}\right\} \text{V. A.}$$



9.2 $y = f(x) = \ln(x^2-6x+8) = \ln\left(\underline{(x-2)(x-4)}\right)$

$$\begin{cases} x-2>0 \\ x-4>0 \end{cases} \Rightarrow \underline{x>4}$$

$$\begin{cases} x-2<0 \\ x-4<0 \end{cases} \Rightarrow \underline{x<2}$$

H.A. $\lim_{x\to\pm\infty} \ln(x^2-6x+8) = \underline{\infty}$

V.A $\lim_{x\to 4^+} \ln(x-2)(x-4) = -\infty \qquad \underline{x=4}$

$\lim_{x\to 2^-} \ln(x-2)(x-4) = -\infty \qquad \underline{x=2}$

10. Find the equation of the slant asymptotes $\lim_{x\to\infty}\left[f(x) - \underbrace{(a+mx)}_{y}\right] = 0$

10.1 $f(x) = \dfrac{-3x^3+2x^2+7x-5}{x^2+x+1} = (-3x+5) + \dfrac{5x-10}{x^2+x+1}$

$$
\begin{array}{r l}
 & -3x+5 \\
x^2+x+1 \,\big) & -3x^3+2x^2+7x-5 \\
\rightarrow & -3x^3-3x^2-3x \\
\hline
 & 0 \quad 5x^2+10x-5 \\
\rightarrow & 5x^2+5x+5 \\
\hline
 & 0 \quad 5x-10
\end{array}
$$

$$\lim_{x\to\infty}\left[f(x) - (-3x+5)\right] = \lim_{x\to\infty}\frac{5x-10}{x^2+x+1} = 0$$

10.2 $f(x) = \sqrt{x^2+2x-3} = \sqrt{(x+1)^2-4}$

$\lim_{x\to\infty}\sqrt{x^2} = \lim_{x\to\infty} x$

$a^2-b^2 = (a+b)(a-b)$

$$f(x) - x = \sqrt{x^2+2x-3} - \sqrt{x^2} = \frac{(x^2+2x-3) - x^2}{\sqrt{x^2+2x-3}+\sqrt{x^2}}$$

$$= \frac{2x-3}{\sqrt{x^2+2x-3}+x}$$

$f(x) - (x+1)$

$= \sqrt{(x+1)^2-4} - \sqrt{(x+1)^2}$

$= \dfrac{\cancel{(x+1)^2}-4-\cancel{(x+1)^2}}{\sqrt{(x+1)^2-4}+x+1}$

$\rightarrow 0$

$$\Rightarrow \lim_{x\to\infty}\left[f(x) - (x+1)\right] = 0 \Rightarrow y = x+1$$

WA #25 (Lesson 25 and Lesson 26)

#2 Find two numbers whose difference is 108 and whose product is a minimum.

$x-y=108$ $\qquad$ $\min xy = \min x(x-108) = f(x)$

$$f'(x) = (x-108)+x = 2x-108 = 0 \Longrightarrow x = 54$$

#1 Find two numbers whose sum is 23 and whose product is a maximum.

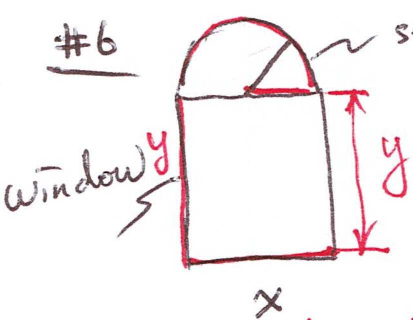
$x+y=23$ $\qquad$ $\max xy = \max x(23-x) = f(x)$

$y = 23-x$ $\qquad$ $f'(x) = (23-x)-x = 23-2x = 0 \Longrightarrow x = \frac{23}{2}$

$$y = \frac{23}{2}.$$

#3 (Lesson 26, #14), **#4** (Lesson 25, Ex. 3), **#5** (Lesson 26, #32)

#6



perimeter $= 32$ ft $= x + 2y + \pi\cdot\frac{x}{2} = x\left(\frac{2+\pi}{2}\right) + 2y$

Find x so that the greatest possible amount of light is admitted.

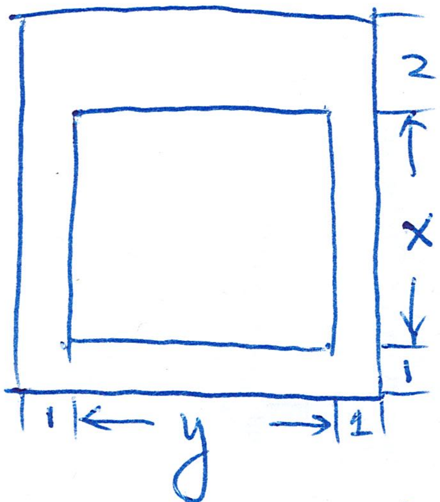
$$\max\left[xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2\right] = \max\left[x\left(16 - \frac{2+\pi}{4}x\right) + \frac{\pi}{8}x^2\right]$$

$$= f(x) = \left(\frac{\pi}{8} - \frac{2+\pi}{4}\right)x^2 + 16x = -\frac{4+\pi}{8}x^2 + 16x$$

$$y = \frac{32 - \frac{(2+\pi)x}{2}}{2} = 16 - \frac{(2+\pi)}{4}x$$

$$f'(x) = -\frac{4+\pi}{8}x + 16 = 0 \Longrightarrow x = \frac{8\cdot 16}{4+\pi}$$

#7 A poster is to have an area of 240 in^2 with 1 inch margins at the bottom and sides and a 2 inch margin at the top. Find the exact dimensions that will give the largest printed area.



$$(y+2)(x+3) = 240 \Rightarrow y = -2 + \frac{240}{x+3}$$

$$\max xy = \max x\left(-2 + \frac{240}{x+3}\right) = \max f(x)$$

$$\cancel{f'(x) = \left(-2 + \frac{240}{x+3}\right) -} \qquad \cancel{f(x) = 240 - 2(x+3) = 234 - 2x}$$

$$f'(x) = \left(-2 + \frac{240}{x+3}\right) - \frac{240x}{(x+3)^2} = \frac{-2(x+3)^2 + 240(x+3) - 240x}{(x+3)^2}$$

$$= \frac{-2(x+3)^2 + 3 \times 240}{(x+3)^2} = 0 \Rightarrow (x+3)^2 = 360$$

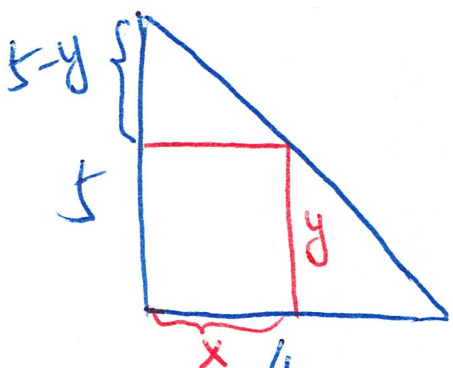
$$x = -3 \pm 6\sqrt{10}$$

$$\Rightarrow x = 6\sqrt{10} - 3, \quad y = -2 + \frac{240}{6\sqrt{10}}.$$

WA #26

#1 (Lesson 26, #26), #3 (Lesson 25, Ex. 4), #4 (Lesson 26, #54)

#2 Find the ~~largest~~ area of the largest rectangle that can be inscribed in a right triangle with legs of length 4 cm and 5 cm if two sides of the rectangle lie along the legs.



$$\frac{x}{4} = \frac{5-y}{5} \Rightarrow x = \frac{4}{5}(5-y)$$

$$\max xy = \max \frac{4}{5}(5-y)y = \max f(y)$$

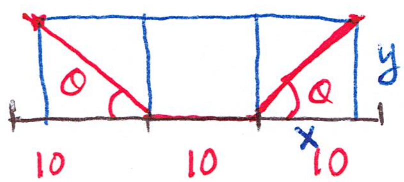
$$f(y) = \frac{4}{5}\left[5y - y^2\right]$$

$$f'(y) = \frac{4}{5}\left[5-2y\right] = 0 \Rightarrow y = \frac{5}{2}$$

$$x = \frac{4}{5}\left(5 - \frac{5}{2}\right) = \frac{4}{5}\cdot\frac{5}{2} = 2$$

$$A = 2\cdot\frac{5}{2} = 5$$

#5 A rain gutter is to be constructed from a metal sheet of ~~width~~ width 30 cm by bending up one-third of the sheet on each side through an angle θ. How should θ be chosen so that the gutter will carry the maximum amount of water?



$$A = 10y + xy$$

$$= 100\left[\sin\theta + \sin\theta\cos\theta\right] = 100\left[\sin\theta + \frac{1}{2}\sin 2\theta\right]$$

$y = 10\sin\theta$

$x = 10\cos\theta$

$$A'(\theta) = 100\left[\cos\theta + \cos 2\theta\right] = 100\left[2\cos^2\theta + \cos\theta - 1\right] = 0$$

$$\Rightarrow 0 = 2\cos^2\theta + \cos\theta - 1$$

$$\Rightarrow \cos\theta = \frac{-1 \pm \sqrt{1^2 - 4\cdot 2\cdot(-1)}}{2\cdot 2} = \frac{-1 \pm \sqrt{9}}{4} = -1 \text{ or } \frac{1}{2}$$

$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ ✓

$\cos\theta = -1 \Rightarrow \theta = \pi$