MA 174 EXAM II 04/04/2014

Name: ____________ PUID: ____________ Section: ____________

SHOW ALL YOUR WORK. NO CALCULATORS, BOOKS, OR PAPERS ARE ALLOWED.

Points awarded

1. (10 points) ____________
2. (10 points) ____________
3. (10 points) ____________
4. (10 points) ____________
5. (15 points) ____________
6. (10 points) ____________
7. (10 points) ____________
8. (10 points) ____________
9. (15 points) ____________

| | | | | | |
|---|---|---|---|---|---|
| 88 | 79 | 67 | 57 | 49 | |
| 86 | 77 | 66 | 53 | | |
| 85 | 71 | 63 | 51 | | |
| 83 | | | 50 | | |
| 81 | | | | | |
| 81 | | | | | |
| 81 | | | | | |
| 81 | | | | | |
| 8 | 3 | 3 | 4 | 1 | =19 |

Total Points: ____________

**1.** (10 points) Find an equation for the tangent plane of the surface

$$f(x,y,z) = xe^y + \cos(xy) + y - z^2 = ln2$$

at point $(0, ln2, 1)$.

$$\nabla f(0, \ln 2, 1) = \left(e^y - y\sin(xy),\ xe^y - x\sin(xy) + 1,\ -2z\right)\Big|_{(0,\ln 2,1)}$$

$$= (2, 1, -2)$$

$$(2, 1, -2) \cdot (x,\ y - \ln 2,\ z - 1) = 0$$

**2.** (10 points) Find $(\frac{\partial w}{\partial x})_y$ at point $(x, y, z) = (0, 1, \pi)$ if

$w = x^2 + y^2 + z^2$ and $y \sin z + z \sin x = 0.$

$$\left(\frac{\partial w}{\partial x}\right)_y = 2x + 2z\frac{\partial z}{\partial x}$$

$$0 = \frac{\partial}{\partial x}\left(y\sin z + z\sin x\right) = y\cos z\frac{\partial z}{\partial x} + \sin x\frac{\partial z}{\partial x} + z\cos x$$

$$\Rightarrow \frac{\partial z}{\partial x}(0,1,\pi) = -\frac{z\cos x}{y\cos z + \sin x}\Big|_{(0,1,\pi)} = -\frac{\pi}{-1} = \pi$$

$$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_y(0,1,\pi) = 2\pi \cdot \pi = 2\pi^2$$

3. (10 points) Use Taylor's formula to find a quadratic approximation of $f(x, y) = e^{2x-y}$ at $(0, 0)$.

$f(0,0) = 1$

$\frac{\partial f}{\partial x}(0,0) = 2e^{2x-y}\Big|_{(0,0)} = 2$, $\frac{\partial f}{\partial y}(0,0) = -e^{2x-y}\Big|_{(0,0)} = -1$

$\frac{\partial^2 f}{\partial x^2}(0,0) = 4e^{2x-y}\Big|_{(0,0)} = 4$, $\frac{\partial^2 f}{\partial y^2}(0,0) = e^{2x-y}\Big|_{(0,0)} = 1$, $\frac{\partial^2 f}{\partial x \partial y}(0,0) = -2$

$$T_2(x,y) = f(0,0) + f_x(0,0)\,x + f_y(0,0)\,y + \frac{1}{2} f_{xx}(0,0)\,x^2 + f_{xy}(0,0)\,xy + \frac{1}{2} f_{yy}(0,0)\,y^2$$

$$= 1 + 2x - y + 2x^2 - 2xy + \frac{1}{2}y^2$$

4. (10 points) If the derivative of $f(x, y)$ at a point $P$ in the direction of $\mathbf{i} + \mathbf{j}$ is $3\sqrt{2}$ and in the direction of $\mathbf{i} - \mathbf{j}$ is $2\sqrt{2}$, what is the gradient of $f(x, y)$ at the point $P$?

$$\left(\frac{df}{ds}\right)_{\vec{i}+\vec{j},P} = \nabla f(P) \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = 3\sqrt{2}$$

$$\left(\frac{df}{ds}\right)_{\vec{i}-\vec{j},P} = \nabla f(P) \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = 2\sqrt{2}$$

$$\Rightarrow \begin{cases} f_x(P) + f_y(P) = 6 \\ f_x(P) - f_y(P) = 4 \end{cases}$$

$$\Rightarrow f_x(P) = 5,\ f_y(P) = 1 \Rightarrow \nabla f(P) = (5, 1)$$

5. (15 points) Find all critical points of function

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

and identify each as a local maximum, local minimum or saddle point.

$$\nabla f = (4x^3 - 4y,\ 4y^3 - 4x) = (0,0)$$

$$\Rightarrow \left\{\begin{array}{l} x^3 - y = 0 \Rightarrow y = x^3 \\ y^3 - x = 0 \end{array}\right\} \Rightarrow 0 = x^9 - x = x(x^8 - 1)$$

$$= x(x^4+1)(x^4-1)$$

$$= x(x^4+1)(x^2+1)(x^2-1)$$

$$\Rightarrow (0,0),\ (1,1),\ (-1,-1) \text{ critical pts}$$

$$f_{xx} = 12x^2,\quad f_{yy} = 12y^2,\quad f_{xy} = -4,\quad D = f_{xx}f_{yy} - f_{xy}^2$$

$\underline{(0,0)}$ $f_{xx}(0,0) = 0,\ f_{yy}(0,0) = 0,\ f_{xy}(0,0) = -4$

$D = 0 - 16 = -16 < 0$ $\underline{\text{saddle pt.}}$

$\underline{(1,1)}$ $f_{xx}(1,1) = 12,\ f_{yy}(1,1) = 12,\ D = 12^2 - (-4)^2 > 0,$ $\underline{\text{l. min}}$

$\underline{(-1,-1)}$ $f_{xx}(-1,-1) = 12,\ f_{yy}(-1,-1) = 12,\ D = 12^2 - (-4)^2 > 0,$ $\underline{\text{l. min}}$

6. (10 points) Computing the following limits

$$(i) \lim_{(x,y)\to(1,1)} \frac{x-y}{x^4-y^4} \qquad\qquad (ii) \lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$$

$$= \lim \frac{x-y}{(x^2+y^2)(x^2-y^2)}$$

$$= \lim \frac{1}{(x^2+y^2)(x+y)}$$

$$= \frac{1}{2\cdot 2} = \frac{1}{4}$$

let $y = \alpha x$, then

$$\lim \frac{x-y}{x+y} = \lim \frac{(1-\alpha)x}{(1+\alpha)x}$$

$$= \frac{1-\alpha}{1+\alpha} = \begin{cases} 1 & \alpha = 0 \\ 0 & \alpha = 1 \end{cases}$$

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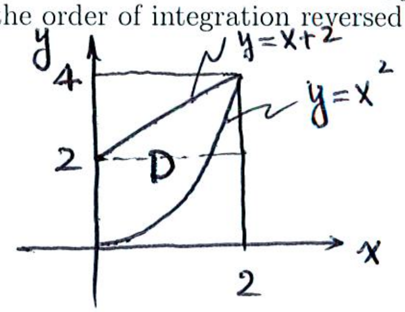
7. (10 points) Sketch the region of the integration for the integral $\int_0^2 \int_{x^2}^{x+2} f(x,y)\,dy\,dx$ and write an equivalent integral with the order of integration reversed.

$$\begin{cases} x^2 \leq y \leq x+2 \\ 0 \leq x \leq 2 \end{cases}$$



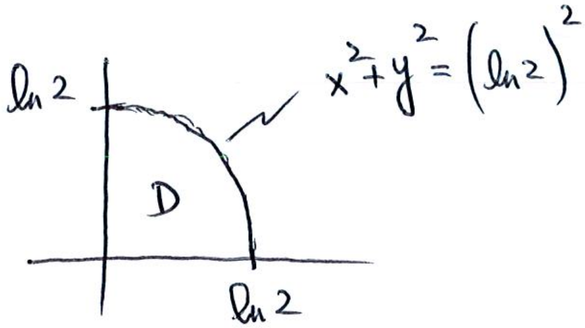

$$\begin{cases} 0 \leq y \leq 2 \\ 0 \leq x \leq \sqrt{y} \end{cases}$$

$$\begin{cases} 2 \leq y \leq 4 \\ y-2 \leq x \leq \sqrt{y} \end{cases}$$

$$\int_0^2 \int_{x^2}^{x+2} f\,dy\,dx = \int_0^2 \int_0^{\sqrt{y}} f\,dx\,dy + \int_2^4 \int_{y-2}^{\sqrt{y}} f\,dx\,dy$$

8. (10 points) Evaluate $\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy$.

$$\begin{cases} 0 \le x \le \sqrt{(\ln 2)^2 - y^2} \\ 0 \le y \le \ln 2 \end{cases}$$




$$\begin{cases} 0 \le r \le \ln 2 \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\ln 2} e^r\, r\, dr\, d\theta \quad 4$$

$$\begin{cases} u = r, & v' = e^r \\ u' = 1, & v = e^r \end{cases}$$

$$= \frac{\pi}{2} \int_0^{\ln 2} e^r r\, dr$$

$$= \frac{\pi}{2} \left[ r e^r \Big|_0^{\ln 2} - \int_0^{\ln 2} e^r\, dr \right]$$

$$= \frac{\pi}{2} \left[ 2 \ln 2 - e^r \Big|_0^{\ln 2} \right] \quad 4$$

$$= \frac{\pi}{2} \left[ 2 \ln 2 - 1 \right] \quad 2$$

**9.** (15 points) Find the greatest and smallest values that the function $f(x, y) = x^2 + y^2 + xy$ takes on the disc $x^2 + y^2 \le 1$.

Inside the disc: $\{x^2+y^2<1\}$

$\nabla f = (2x+y,\ 2y+x) = (0,0) \Rightarrow \begin{cases} 2x+y=0 \\ 2y+x=0 \end{cases} \Rightarrow (x,y)=(0,0)$

critical pt $(0,0)$ and $f(0,0)=0$. — min 3

On the circle: $\{x^2+y^2=1\}$ let $g(x,y)=x^2+y^2$

$$\min_{x^2+y^2=1}/\max\left(x^2+y^2+xy\right) = \min_{\underset{g(x,y)}{\underset{\parallel}{x^2+y^2=1}}}/\max\ (\underset{h(x,y)}{\underset{\parallel}{1+xy}})$$

Critical pts $\nabla h = (y, x)$, $\nabla g = (2x, 2y)$

$$\begin{cases} \nabla h = \lambda \nabla g \\ g=1 \end{cases} \Rightarrow \begin{cases} y = 2x\lambda \\ x = 2y\lambda \\ x^2+y^2=1 \end{cases} \Rightarrow x = 4x\lambda^2$$

$x(4\lambda^2-1)=0$

$x=0$ or $\lambda = \pm\frac{1}{2}$

$x=0 \Rightarrow y=0$ but $0^2+0^2 \ne 1$

$\lambda = \pm\frac{1}{2} \Rightarrow y = \pm x \overset{x^2+y^2=1}{\Longrightarrow} 2x^2=1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

critical pts $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{3}{2}$, $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$

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