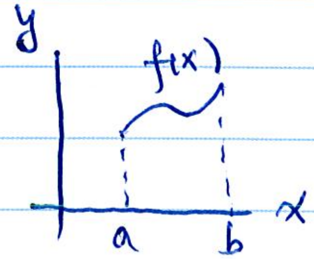
Chapter 13 Vector-Valued Functions and Motion in Space

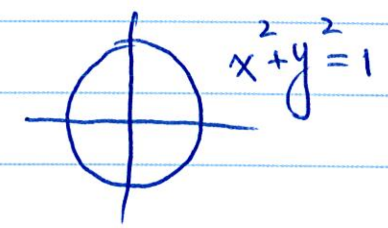
§13.1 Curves in Space and Their Tangents

Representations of a curve in R^2



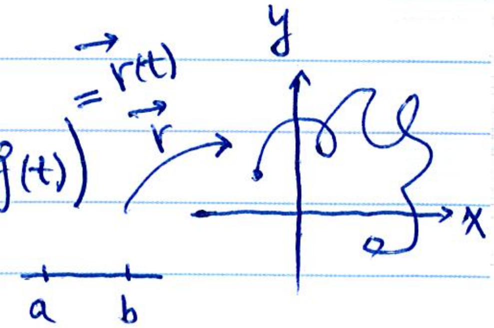
- graph $\quad y = f(x) \quad x \in [a,b]$

$$C = \{ (x, f(x)) \mid x \in [a,b] \}$$



- level curve $\quad f(x,y) = k$ – const.

$$C = \{ f(x,y) = k \mid (x,y) \in A \}$$



- parametric curve $\quad (x, y) = (f(t), g(t)) = \vec{r}(t)$

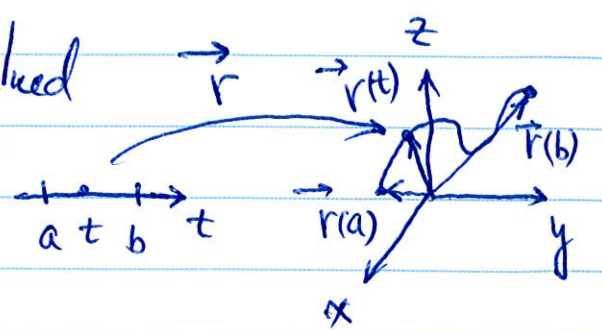
$$C = \{ (f(t), g(t)) \mid t \in [a,b] \}$$

Path in R^3 $\quad (x, y, z) = (f(t), g(t), z(t)) \quad t \in I = [a,b]$

$= \vec{r}(t)$ – vector-valued function

parametric curve

$$C = \{ (f(t), g(t), z(t)) \mid t \in I \}$$



Ex. 1 Graph $\vec{r}(t) = (\cos t, \sin t, t)$

Limits and Continuity

$\vec{r}(t) = (f(t), g(t), h(t))$ on D and $\vec{L} = (L_1, L_2, L_3)$

- **Limit**

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L} \iff \forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } \forall t \in D$$

$$\text{"} 0 < |t - t_0| < \delta \Rightarrow |\vec{r}(t) - \vec{L}| < \varepsilon \text{"}$$

$$\iff \lim_{t \to t_0} \vec{r}(t) = \left(\lim f(t), \lim g(t), \lim h(t)\right)$$

$$= (L_1, L_2, L_3)$$

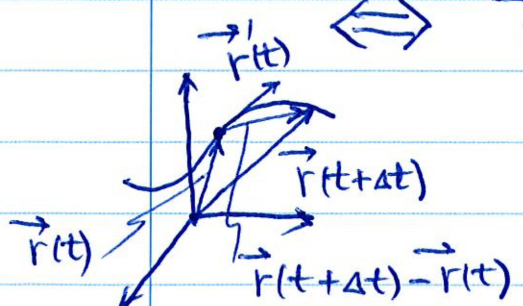
Ex. 2 $\displaystyle \lim_{t \to \frac{\pi}{2}} (\cos t, \sin t, t) = ?$

- **Continuity** $\vec{r}(t)$ is cont. at $t_0 \in D \iff \displaystyle\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$

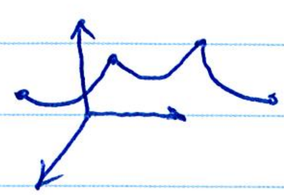
Derivatives and Motion

- **Derivative** $\vec{r}(t)$ has a derivative (is differentiable) at t

$$\iff \vec{r}\,'(t) = \frac{d\vec{r}(t)}{dt} = (f'(t), g'(t), h'(t))$$



piecewise smooth



Velocity and Acceleration

$\vec{r}(t)$ — position vector of a moving particle

$\vec{v}(t) = \vec{r}\,'(t)$ — velocity vector

$|\vec{v}(t)|$ — speed

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$ — acceleration

$\dfrac{\vec{v}(t)}{|\vec{v}(t)|}$ — direction of motion

Ex. 4

Differential Rules

(1) $\vec{c}$ — const, $\vec{c}\,' = \vec{0}$, (2) $\left(f(t)\,\vec{u}(t)\right)' = f'\vec{u} + f\vec{u}\,'$

(3) $\left(\vec{u} \pm \vec{v}\right)' = \vec{u}\,' \pm \vec{v}\,'$, (4) $\left(\vec{u}\cdot\vec{v}\right)' = \vec{u}\,'\cdot\vec{v} + \vec{u}\cdot\vec{v}\,'$

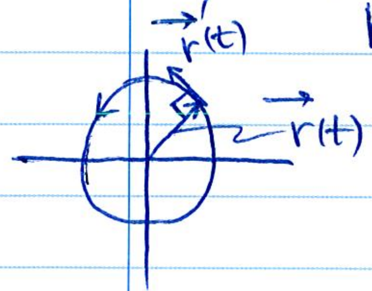
(5) $\left(\vec{u}\times\vec{v}\right)' = \vec{u}\,'\times\vec{v} + \vec{u}\times\vec{v}\,'$, (6) $\dfrac{d}{dt}\vec{u}(f(t)) = \vec{u}\,'(f(t))\,f'(t)$

Proof of (4)

Proof of (5) $\left(\vec{u}\times\vec{v}\right)' = \lim\limits_{t\to 0}\dfrac{1}{h}\left[\boxed{\vec{u}(t+h)\times\vec{v}(t+h) \quad -\vec{u}(t)\times\vec{v}(t+h)} - \boxed{\vec{u}(t)\times\vec{v}(t) \quad +\vec{u}(t)\times\vec{v}(t+h)}\right]$

Vector Functions of Const Length $\quad |\vec{r}(t)| = \text{const.}$

$$\vec{r}(t)\cdot\vec{r}\,'(t)=0$$



$$c=\vec{r}\cdot\vec{r} \Rightarrow 0=\left(\vec{r}\cdot\vec{r}\right)' = 2\,\vec{r}\cdot\vec{r}\,'.$$

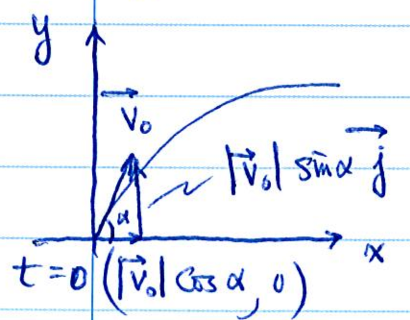
§13.2 Integrals of Vector Functions; Projectile Motion

$$\int \vec{r}(t)\,dt = \left(\int f(t)\,dt,\ \int g(t)\,dt,\ \int h(t)\,dt\right)$$

Ex. 1, 2, 3

A classical example ~~of~~ is the derivation of the projectile motion.

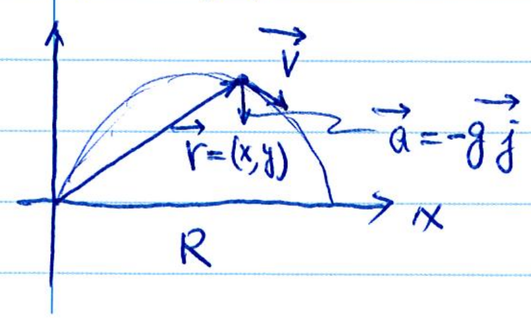
Ideal Projectile Motion



$$\vec{v}_0 = \left(|\vec{v}_0|\cos\alpha,\ |\vec{v}_0|\sin\alpha\right)$$

$$= \left(v_0\cos\alpha,\ v_0\sin\alpha\right)$$

$$\vec{r}_0 = (0,0)$$



$$m\,\vec{r}\,''(t) = -mg\,\vec{j}$$

$$\Rightarrow \vec{r}\,'(t) = -(gt)\,\vec{j} + \vec{v}_0$$

$$\Rightarrow \vec{r}(t) = -\tfrac{1}{2}gt^2\,\vec{j} + \vec{v}_0 t + \vec{r}_0$$

$$\vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + v_0 t\cos\alpha\,\vec{i} + v_0 t\sin\alpha\,\vec{j}$$

$$\begin{cases} x(t) = v_0 t\cos\alpha \\ y(t) = v_0 t\sin\alpha - \frac{1}{2}gt^2 \end{cases}$$

$$t = \frac{x}{v_0\cos\alpha} \Rightarrow y = \underbrace{-\left(\frac{g}{2v_0^2\cos^2\alpha}\right)x^2 + (\tan\alpha)x}_{\text{parabola}}$$

Maximum height $\quad y_{max} = \frac{(v_0\sin\alpha)^2}{2g} \Leftarrow 0 = \frac{dy}{dx}$

Flight time $\quad t = \frac{2v_0\sin\alpha}{g} \Leftarrow y = 0 = v_0 t\sin\alpha - \frac{1}{2}gt^2$

Range $\quad R = \frac{v_0^2}{g}\sin 2\alpha \Leftarrow x(\text{flight time}) = R$.

Ex. 4 Given $\vec{r}_0 = (0,0)$, $v_0 = 500\ m/sec$, $\alpha = 60^\circ$

Find $\vec{r}(10) = ?$

Projectile Motion with Wind Gusts — $a = 8.8\ ft/sec$

Ex. 5 Given $\vec{v}_0 = (v_0\cos\alpha, v_0\sin\alpha) - (a, 0)$

$\vec{r}_0 = (0,3)$, $v_0 = 152\ ft/sec$, $\alpha = 20^\circ$

Find (a) $\vec{r}(t) = ?$

(b) $y_{max} = ?$

(c) $R = ?$ Flight time $= ?$

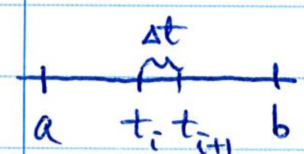
§13.3 Arc Length in Space

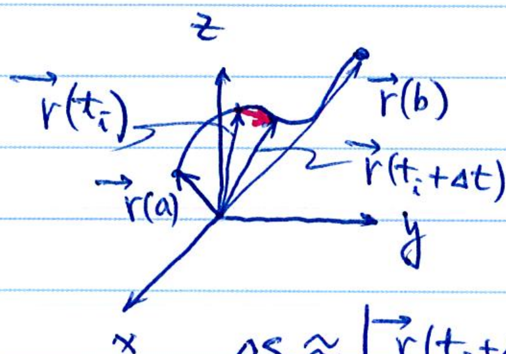
$$\vec{r}(t) = \big(x(t), y(t), z(t)\big) \qquad t \in [a,b] \quad \underline{\text{traced exactly once}}$$

Length

$$L = \int_a^b |\vec{v}|\,dt$$

$$= \int_a^b \sqrt{x'^2 + y'^2 + z'^2}\,dt = \int_C ds$$





$$\Delta s_i \approx \left|\vec{r}(t_i+\Delta t) - \vec{r}(t_i)\right|$$

$$L = \lim \sum_i \left|\vec{r}(t_i+\Delta t) - \vec{r}(t_i)\right|$$

Ex. 1

Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\,d\tau$$

Ex. 2

Speed on a Smooth Curve

speed $\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t} |\vec{v}(\tau)|\,d\tau = |\vec{v}(t)| > 0$

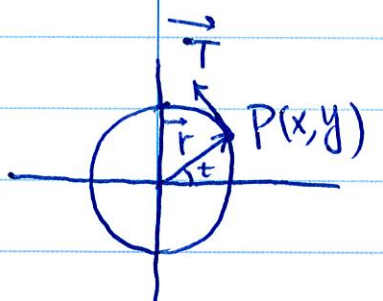
$\Longrightarrow$ arclength s is an increasing function of t

Unit Tagent Vector

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$ tangent to the curve

Ex. 3

~~Ex.~~



$\vec{r} = (\cos t, \sin t)$

$\vec{v} = (-\sin t, \cos t) \Longrightarrow \vec{T} = \vec{v}$

$\quad \parallel$

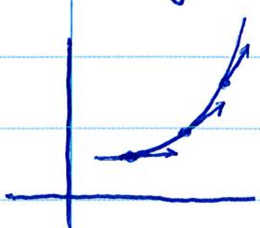
$\frac{d\vec{r}}{dt}$

in general $\quad ? = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\cdot\left(\frac{dt}{ds}\right) = \vec{v}\left(\frac{1}{\frac{ds}{dt}}\right) = \frac{\vec{v}}{|\vec{v}|} = \vec{T}(t)$

§13.4 Curvature and Normal Vector of a Curve

How a curve turns or bends

Curvature of a Plane Curve



As a particle moves along a smooth curve in the plane

$\vec{T} = \frac{d\vec{r}}{ds}$ turns as the curve bends.

unit tangent vector

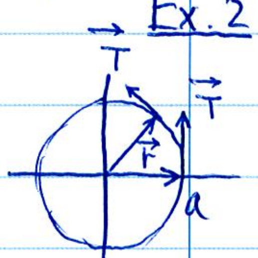
curvature $\boxed{\kappa = \left|\frac{d\vec{T}}{ds}\right|}$ the rate ~~of~~ in which $\vec{T}$ turns per unit length along the curve

$$\boxed{\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|} = \left|\frac{d\vec{T}}{dt}\cdot\frac{dt}{ds}\right| = \left|\frac{d\vec{T}}{dt}\right|\frac{1}{\left|\frac{ds}{dt}\right|} = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

<u>Ex. 1</u> $\vec{r}(t) = \underbrace{\vec{C}}_{\text{const.}} + t\vec{v}$ $\Rightarrow \vec{r}'(t) = \vec{v}$ and $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ const

$$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = 0$$

<u>Ex. 2</u> $\vec{r}(t) = (a\cos t, a\sin t)$ $t \in [0, 2\pi]$



$$\kappa \overset{?}{=} \frac{1}{a}$$

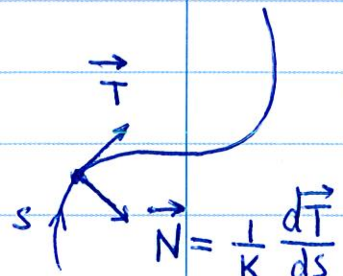
$$s(t) = \int_0^t |\vec{r}'(\tau)|\,d\tau = at \Rightarrow \frac{ds}{dt} = a$$

$$\vec{T} = \vec{r}'(t)/|\vec{r}'(t)| = \frac{1}{a}(-a\sin t, a\cos t) = (-\sin t, \cos t) \qquad \vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = -\frac{\vec{r}(t)}{a} \perp \vec{T}$$

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\cdot\frac{dt}{ds} = \frac{1}{a}\frac{d\vec{T}}{dt} = \frac{-1}{a}(\cos t, \sin t) = -\frac{1}{a^2}\vec{r}(t)$$

Principal Unit Normal Vector at $\kappa \neq 0$

$$\boxed{\cancel{\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}}}$$



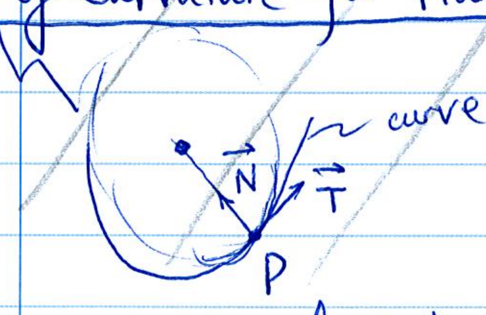
$$|\vec{T}| = 1 \Rightarrow \vec{T}(s)\cdot\vec{T}'(s) = 0 \Longleftrightarrow \vec{T}(s) \perp \frac{d\vec{T}}{ds}$$

$$\underbrace{\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}}_{\text{definition}} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

Ex. 3 $\vec{r}(t) = (\cos 2t, \sin 2t)$

$\vec{T} = ?$ $\vec{N} = ?$

Circle of Curvature for Plane Curves (see p749)



Curvature and Normal Vectors in Space Curves

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|, \quad \vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

Ex. 5 $\vec{r}(t) = (a\cos t, a\sin t, bt)$, $a, b \geq 0$, $a^2 + b^2 \neq 0$

$\kappa = ?$ $\vec{N} = ?$

§13.5 Tangential and Normal Components of Acceleration

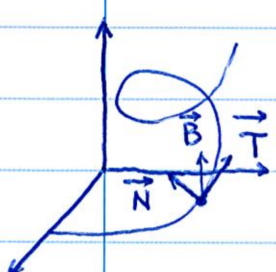
traveling along a space curve

- Cartesian $\vec{i}, \vec{j}, \vec{k}$ — not truly relevent
- TNB frame or Frenet frame

$\vec{T} = \dfrac{\vec{v}}{|\vec{v}|}$ forward moving direction

$\vec{N} = \dfrac{1}{\kappa}\dfrac{d\vec{T}}{ds}$ turning direction

binormal vector $\vec{B} = \vec{T} \times \vec{N}$ tendency to "twist" out of the plane ~~spa~~ span$\{\vec{T}, \vec{N}\}$



- tangential and normal components of acceleration

$$\vec{a} = a_T\,\vec{T} + a_N\,\vec{N}$$

$a_T = \dfrac{d^2s}{dt} = \dfrac{d}{dt}|\vec{v}|$ tangential components of acceleration

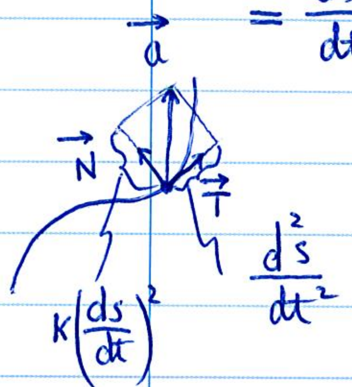
$a_N = \kappa\left(\dfrac{ds}{dt}\right)^2 = \cancel{\kappa(\ )^2}\ \kappa|\vec{v}|$ normal component

$$\vec{a} = \frac{d}{dt}\vec{v} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \vec{T}\frac{ds}{dt}$$

$$= \frac{d}{dt}\left(\vec{T}\frac{ds}{dt}\right)$$

$$= \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\cdot\frac{d\vec{T}}{dt} \qquad \frac{d\vec{T}}{ds} = \kappa\vec{N}.$$

$$= \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2\frac{d\vec{T}}{ds} = \frac{d^2s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2\vec{N}$$



$$\boxed{a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \sqrt{|\vec{a}|^2 - a_T^2}}$$

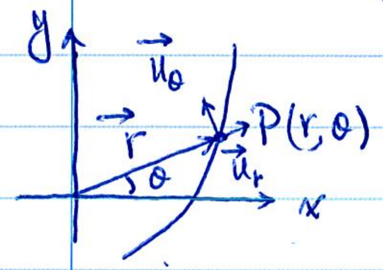
Ex. 1 $\quad \vec{r}(t) = \left(\cos t + t\sin t,\ \sin t - t\cos t\right) \qquad t > 0$

$\vec{a} = a_T\vec{T} + a_N\vec{N} = ?$ without computing $\vec{T}$ & $\vec{N}$

$a_T = \frac{d}{dt}|\vec{v}|, \qquad a_N = \sqrt{|\vec{a}|^2 - a_T^2}$

§13.6 Velocity and Acceleration in Polar Coordinates

Motion in Polar and Cylindrical Coordinates



$$\vec{u}_r = (\cos\theta, \sin\theta) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$\vec{u}_\theta = (-\sin\theta, \cos\theta) \qquad \vec{u}_r \perp \vec{u}_\theta$$

$$\vec{r} = r\,\vec{u}_r \qquad \|\vec{u}_r\| = \|\vec{u}_\theta\| = 1$$

$$\begin{cases} \dfrac{d\vec{u}_r}{d\theta} = \vec{u}_\theta \\[2ex] \dfrac{d\vec{u}_\theta}{d\theta} = -\vec{u}_r \end{cases} \Longrightarrow \begin{cases} \dfrac{d\vec{u}_r}{dt} = \dfrac{d\vec{u}_r}{d\theta}\cdot\dfrac{d\theta}{dt} = \dot\theta\,\vec{u}_\theta \\[2ex] \dfrac{d\vec{u}_\theta}{dt} = -\dot\theta\,\vec{u}_r \end{cases}$$

$$\vec{v} = \dot{\vec{r}} = \dot r\,\vec{u}_r + r\dot\theta\,\vec{u}_\theta$$

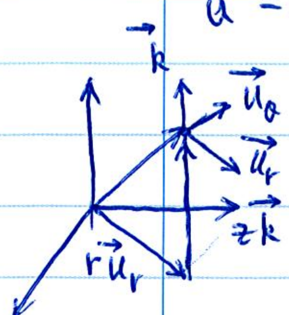
$$\vec{a} = \dot{\vec{v}} = (\ddot r - r\dot\theta^2)\vec{u}_r + \boxed{\cancel{\dot r\dot\theta\,\vec{u}_\theta + r\ddot\theta\,\vec{u}_\theta + r\dot\theta\,\dot{\vec{u}}_\theta}}\,(r\ddot\theta + 2\dot r\dot\theta)\vec{u}_\theta$$

R^3

$$\vec{r} = r\,\vec{u}_r + z\vec{k}$$

$$\vec{v} = \dot r\,\vec{u}_r + r\dot\theta\,\vec{u}_\theta + \dot z\,\vec{k}$$

$$\vec{a} = (\ddot r - r\dot\theta^2)\vec{u}_r + (r\ddot\theta + 2\dot r\dot\theta)\vec{u}_\theta + \ddot z\,\vec{k}$$

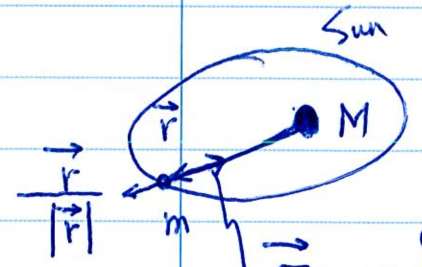


$$\vec{u}_r \times \vec{u}_\theta = \vec{k}, \quad \vec{u}_\theta \times \vec{k} = \vec{u}_r, \quad \vec{k} \times \vec{u}_r = \vec{u}_\theta$$

Planets Move in Planes

$$G \approx 6.6726 \times 10^{-11} \ \mathrm{N m^2 kg^{-2}}$$

universal gravitational constant



$$\vec{F} = -\frac{GmM}{|\vec{r}|^2} \cdot \frac{\vec{r}}{|\vec{r}|}$$

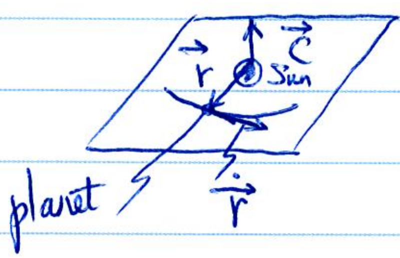
$$\vec{F} = m\vec{a} \implies \ddot{\vec{r}} = -\frac{GmM}{|\vec{r}|^3}\vec{r} \implies \ddot{\vec{r}} \times \vec{r} = \vec{0}$$

$$\frac{d}{dt}\left(\vec{r} \times \dot{\vec{r}}\right) = \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}} = \vec{0}$$

$$\Downarrow$$

$\vec{r} \times \dot{\vec{r}} = \vec{c}$ — const. $\implies$ $\vec{r}$ & $\dot{\vec{r}}$ lie in a plane $\perp \vec{c}$

$\implies$ the planet moves in a fixed plane



Kepler's First Law (Ellipse Law)

$$r = |\vec{r}|$$

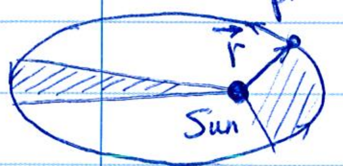
$$r = \frac{(1+e)\,r_0}{1 + e\cos\theta}$$ — ellipse equation in polar coordinates

where $e = \dfrac{r_0 v_0^2}{GM} - 1$ eccentricity

r_0 — min distance

v_0 — speed at min distance

<u>Kepler's 2nd Law</u> the radius vector from the sun to a planet $\vec{r}$ sweeps out equal area in equal times.



$$\vec{C} = \vec{r} \times \dot{\vec{r}} = r\left(r\dot{\theta}\right)\vec{k}$$

$$= \left[r\left(r\dot{\theta}\right)\right]_{t=0}\vec{k} = r_0 v_0 \vec{k}$$

$$\Longrightarrow \quad r_0 v_0 \vec{k} = r^2 \dot{\theta}\vec{k} \quad \Longrightarrow \quad r^2\dot{\theta} = r_0 v_0$$

$$dA = \frac{1}{2} r^2 d\theta \quad \Longrightarrow \quad \frac{dA}{dt} = \frac{1}{2} r_0 v_0 = \text{const}$$

(Section 11.5)

$\cancel{\Longrightarrow}$

<u>Kepler's 3rd Law</u>

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$

T — the planet's orbital period

a — semi major axis