Chapter 14 Partial Derivatives

§14.1 Functions of Several Variables

$w = f(x_1, \cdots, x_n)$ $\quad (x_1, \cdots, x_n) \in D$ — domain

dep. variable $\quad$ indep. variable

range

	Domain	Range
Example $z = \sqrt{y - x^2}$		
$z = \dfrac{1}{xy}$		
$z = \sin(xy)$		
$w = \sqrt{x^2+y^2+z^2}$		
$w = \dfrac{1}{x^2+y^2+z^2}$		
$w = xy \ln z$		

Region R

interior pts

boundary pts

open region

close region

R is bounded

unbounded

Grap, Level Set

$$\underline{\text{Graph}} = \left\{ (x, y, z, f(x,y,z)) \,\middle|\, (x,y,z) \in D \right\}$$

$$\text{Level set} = \left\{ (x,y,z) \,\middle|\, f(x,y,z) = k \leftarrow \text{const} \right\}$$

$$\text{contour set} = \left\{ (x,y,z,k) \,\middle|\, f(x,y,z) = k \right\}$$

§14.2 Limits and Continuity

Definition $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L \Longleftrightarrow$ $\forall \varepsilon>0, \exists \delta>0$, s.t. $0<|(x,y)-(x_0,y_0)|<\delta \Rightarrow |f(x,y)-L|<\varepsilon$

$\lim\limits_{(x,y)\to(x_0,y_0)} x = x_0$, $\lim k = k$ | Proof

$\lim y = y_0$,

Properties 1–7 (p775)

Ex. 1, 2, 3, 4.

Continuity $f(x,y)$ is cont. at (x_0, y_0)

$\Longleftrightarrow \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Ex. 5

Two-Path Test for Nonexistence of a Limit

2 different paths lead to 2 diff. limit $\Longrightarrow \lim f$ DNE.

Ex. 6

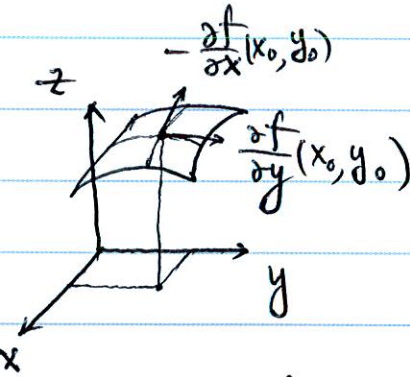
§14.3 Partial Derivatives

$z = f(x, y)$

Definition $\dfrac{\partial f}{\partial x}(x_0, y_0) = \lim\limits_{h \to 0} \dfrac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$, $\dfrac{\partial f}{\partial y}(x_0, y_0) = \cdots$

$\dfrac{\partial f}{\partial x} = \dfrac{\partial z}{\partial x} = f_x$

Geometric Interpretation



Ex. (1) $f(x,y) = x^2 + 3xy + y - 1$, $\dfrac{\partial f}{\partial x}(4,-5) = ?$, $\dfrac{\partial f}{\partial y}(4,-5) = ?$

(2) $f = y \sin xy$, $\dfrac{\partial f}{\partial y} = ?$

(3) $f = \dfrac{2y}{y + \cos x}$, $f_x = ?$, $f_y = ?$

(4) $yz - \ln z = x + y$, $\dfrac{\partial z}{\partial x} = ?$

Partial Der. and Continuity

Ex. 8 $f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

- $\lim\limits_{\substack{(x,y)\to(0,0) \\ y = x}} f(x,y) = \lim 0 = 0$
- f is not cont. at $(0,0)$
- $\dfrac{\partial f}{\partial x}(0,0) = 0 = \dfrac{\partial f}{\partial y}(0,0)$

2^{nd}-Order Partial Der.

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^3 f}{\partial x \partial y^2}, \quad \ldots$$

Ex. 11 $f = 1 - 2xy^2z + x^2y$, $\quad f_{yxyz} = ?$

Mixed Der. Thrm

Assumptions $f, f_x, f_y, f_{yx}, f_{xy}$ — ~~cont~~ defined in an open region containing (a,b)

cont. at (a,b)

$$\Longrightarrow \quad f_{yx}(a,b) = f_{xy}(a,b)$$

Differentiability f is diff. at (x_0, y_0)

$$\Longleftrightarrow \quad \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \left[f(x_0,y_0) + \nabla f(x_0,y_0)\cdot\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}\right]}{\left|(x,y)-(x_0,y_0)\right|} = 0$$

Thrm f_x and f_y are cont. at an open region containing (x_0, y_0)

$\Longrightarrow$ f is diff. at (x_0, y_0).

Thrm f is diff at $(x_0,y_0) \Longrightarrow f$ is cont. at (x_0, y_0).

§14.4 The Chain Rule

$$w = f(x), \quad x = g(t) \Longrightarrow w = f\big(g(t)\big)$$

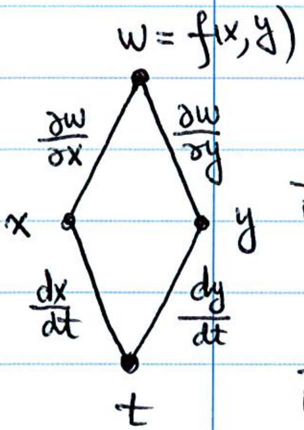
$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt} = f'(g(t))\,g'(t)$$

Functions of 2 Variables

$$w = f(x,y), \quad \begin{matrix} x = x(t) \\ y = y(t) \end{matrix} \Longrightarrow w = f\big(x(t), y(t)\big)$$

$$\frac{dw}{dt} = f_x\big(x(t), y(t)\big)\,x'(t) + f_y\big(x(t), y(t)\big)\,y'(t)$$

$$= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$



dep. v.

intermediate v.

indep. variable

<u>Ex. 1</u> $w = xy$, $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$

$$\left.\frac{dw}{dt}\right|_{t=\frac{\pi}{2}} = ?$$

Functions of 3 Variables

$$w = f(x,y,z), \quad \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \Longrightarrow w = f\big(x(t), y(t), z(t)\big)$$

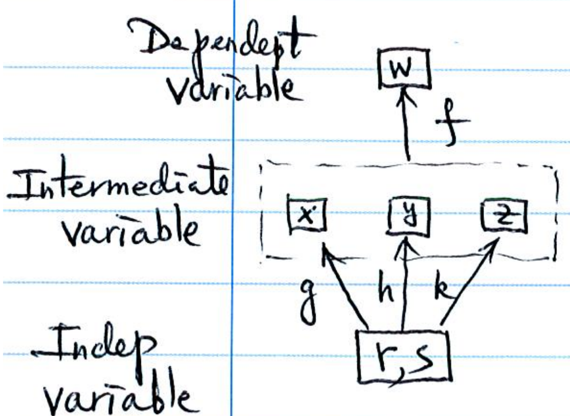
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

<u>Ex. 2</u> $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$, $\frac{dw}{dt} = ?$

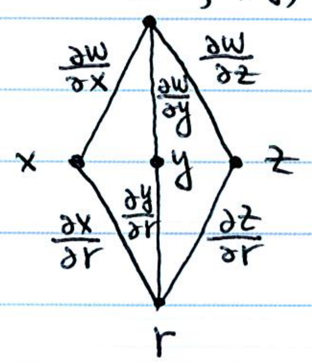
Functions defined on Surfaces

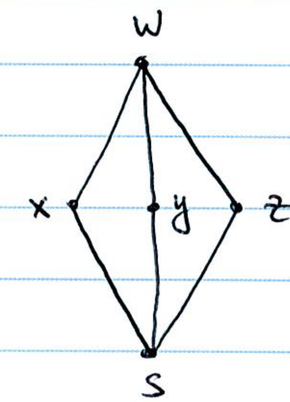
$$w=f(x,y,z) \quad \begin{cases} x=g(r,s) \\ y=h(r,s) \\ z=k(r,s) \end{cases} \Longrightarrow w=f\big(g(r,s),h(r,s),k(r,s)\big)$$

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}, \quad \frac{\partial w}{\partial s}=\cdots$$



$w=f(x,y,z)$





Ex. 3 $w=x+2y+z^2$ $\begin{cases} x=\frac{r}{s} \\ y=r^2+\ln s \\ z=2r \end{cases}$ $\frac{\partial w}{\partial r}=?$ $\frac{\partial w}{\partial s}=?$

Ex. 4 $W=x^2+y^2$, $\begin{cases} x=r-s \\ y=r+s \end{cases}$, $\frac{\partial w}{\partial r}=?$ $\frac{\partial w}{\partial s}=?$

Implicit Diff. Revisited

$$F(x,y)=0 \Longrightarrow \frac{dy}{dx}=-\frac{F_x}{F_y}$$

Ex. 5 $y^2-x^2-\sin xy=0$
$\frac{dy}{dx}=?$

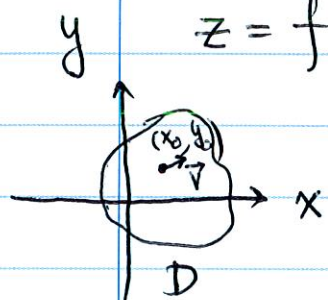
$$F(x,y,z)=0 \Longrightarrow \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

$\Downarrow$

$z=f(x,y)$

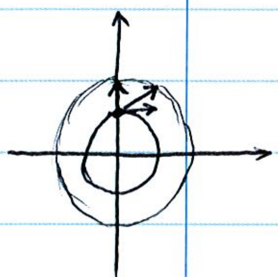
§14.5 Directional Derivatives and Gradient Vectors

$z = f(x,y)\ ,\ (x,y) \in D \subset R^2$



In which direction at (x_0,y_0) does the value of $f(x,y)$ increase most rapidly?

Find $\vec{v} \in R^2$ s.t. $\displaystyle \max_{\substack{\vec{v} \in R^2 \\ |\vec{v}|=1}} \frac{d}{dt} f\left((x_0,y_0)+t\vec{v}\right)\Big|_{t=0}$



$\displaystyle = \max_{\vec{v}} \nabla f(x_0,y_0)\cdot\vec{v}$

$\displaystyle = \max_{\vec{v}} \left|\nabla f(x_0,y_0)\right| \cos\langle \nabla f(x_0,y_0), \vec{v}\rangle$

$f(x,y)=x^2+y^2$

$= \left|\nabla f(x_0,y_0)\right|$ if $\nabla f(x_0,y_0) \,//\, \vec{v}$.

directional derivative along $\vec{u}$

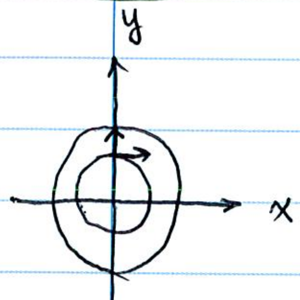
$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \nabla f(x_0,y_0)\cdot\vec{u} = D_{\vec{u}} f(x_0,y_0)$$

gradient vector $\quad \nabla f = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right)$ the direction in which f increases most rapidly.

Ex. 2 $\quad f = xe^y + \cos(xy)\ ,\ \vec{u} = \dfrac{\vec{v}}{|\vec{v}|}\ ,\ \vec{v}=(3,-4)\quad \left(\dfrac{df}{ds}\right)_{\vec{u},(2,0)} = ?$

Ex. 3 $\quad f = \dfrac{x^2}{2} + \dfrac{y^2}{2}$

Gradients and Tangents to Level Curves



$f(x,y) = x^2 + y^2$

Level curve C: $f(x,y) = c$

its parametrization: $\vec{r}(t) = (g(t), h(t))$

$$\Rightarrow \quad f(g(t), h(t)) = c$$

$$\Rightarrow \quad \nabla f \cdot \vec{r}\,'(t) = 0$$

$$\Rightarrow \quad \boxed{\nabla f \perp C = \{(x,y) \mid f(x,y) = c\}}$$

Tangent Equation to $f(x,y) = c$ at $P(x_0, y_0)$

$$\nabla f(x_0, y_0) \cdot (x - x_0,\, y - y_0) = 0$$

Ex. 4 $\quad f(x,y) = \dfrac{x^2}{4} + y^2 = 2$

Find tangent line at $(-2, 1)$

Algebra Rules for Gradients

$$\nabla(f \pm g) = \nabla f \pm \nabla g, \quad \nabla(kf) = k\nabla f, \quad \nabla(fg) = g\nabla f + f\nabla g$$

§14.6 Tangent Planes and Differentials

Tangent Planes and Normal Lines

level surface S $\quad f(x,y,z)=c \qquad$ ~~its param.~~ a smooth curve on S $\quad \vec{r}(t)=\left(g(t),h(t),k(t)\right)$

$$\Rightarrow \quad f\left(\vec{r}(t)\right)=c \;\Rightarrow\; \nabla f\left(\vec{r}(t)\right)\cdot\vec{r}\,'(t)=0$$

$$\Rightarrow \quad \nabla f \perp S=\left\{(x,y,z)\,\middle|\, f(x,y,z)=c\right\}$$

Equation of Tangent Plane at $(x_0,y_0,z_0)\in S$

(1) level surface $\quad f(x,y,z)=c$

$$\nabla f(x_0,y_0,z_0)\cdot\left(x-x_0,\,y-y_0,\,z-z_0\right)=0$$

(2) graph $\qquad z=f(x,y)$

$$g(x,y,z)=f(x,y)-z=0 \;\Longrightarrow\; \nabla g=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},-1\right)$$

$$\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},-1\right)_{(x_0,y_0)}\cdot\left(x-x_0,\,y-y_0,\,z-z_0\right)=0$$

Ex. 1. $f=x^2+y^2+z-9=0$ at $(1,2,4)$

2. $z=x\cos y-ye^{x}$ at $(0,0,0)$

Ex. 3 surfaces $\begin{cases} f = x^2+y^2-2=0 & \text{cylinder} \quad S_1 \\ g = x+z-4=0 & \text{plane} \quad S_2 \end{cases}$

intersection $=$ ellipse E

Find parametric eq. of the line tangent to E at $(1,1,3)$

Solution $\nabla f(1,1,3) \perp S_1$ $\Longrightarrow$ the tangent line $\perp$ $\nabla f(P_0)$ and $\nabla g(P_0)$

$\nabla g(1,1,3) \perp S_2$

$$\Longrightarrow \vec{v} = \nabla f(P_0) \times \nabla g(P_0)$$

$$\Longrightarrow (x,y,z) = (1,1,3) + t\vec{v}.$$

#

Equation of Normal Line at $(x_0, y_0, z_0) \in S$

(1) level surface $f(x,y,z) = c$

$$(x,y,z) = (x_0, y_0, z_0) + t \, \nabla f(x_0, y_0, z_0)$$

(2) graph $z = f(x,y)$

$$(x,y,z) = (x_0, y_0, z_0) + t \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right)_{(x_0, y_0)}$$

Estimating the change in f in a direction $\vec{u}$

one variable $\quad df = f'(P_0)\, ds$

multi-variable $\quad df = \left(\nabla f(P_0) \cdot \vec{u}\right) ds$

Ex. 4 $\quad f = y \sin x + 2yz, \quad ds = 0.1, \quad \vec{u} = \dfrac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|}$

$P_0(0,1,0), \; P_1(2,2,-2), \quad df = ?$

Linearization

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \varepsilon \cdot (x - x_0), \qquad \varepsilon \to 0 \text{ as } x \to x_0$$

$$f(x,y) = \underbrace{f(x_0, y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}_{\substack{\| \\ L(x,y)}} + \underset{\substack{\downarrow \\ 0}}{\varepsilon_1}(x-x_0) + \underset{\substack{\downarrow \\ 0}}{\varepsilon_2}(y-y_0)$$

$L(x,y)$ ~ linear approx.

$$|E(x,y)| = |f(x,y) - L(x,y)| \le \frac{1}{2} M \left(|x-x_0| + |y-y_0| \right)^2$$

$$M = \max_{(x,y)\in R} \left\{ |f_{xx}|, |f_{yy}|, |f_{xy}| \right\}$$

R — a rectangle centered at (x_0, y_0)

Ex. 5, 6 $\quad f(x,y) = x^2 - xy + \frac{1}{2} y^2 + 3$

(1) linearization at $(3,2)$

(2) $|E(x,y)| \le ? \qquad R = \left\{ (x,y) \,\middle|\, |x-3| \le 0.1, \; |y-2| \le 0.1 \right\}$

Differentials

- $y = f(x)$, $\Delta f = f(a+\Delta x) - f(a)$

the differential of f $\quad df = f'(a)\,\Delta x$

- $f(x,y)$, $\Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$, $\Delta x = x - x_0$, $\Delta y = y - y_0$

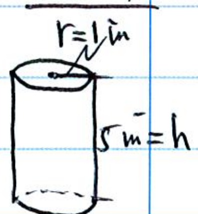
$$\Delta L = L(x_0+\Delta x, y_0+\Delta y) - L(x_0,y_0)$$
$$= f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y$$

the total differential of f

$$df = f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y$$

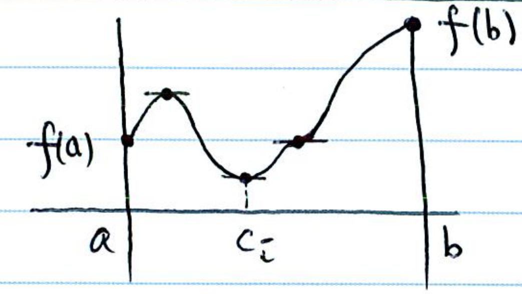
Ex. 7 $\quad dr = 0.03$, $dh = -0.1$, $V = \pi r^2 h$, $r_0 = 1$, $h_0 = 5$

$$\Delta V \approx dV = V_r(r_0,h_0)\,dr + V_h(r_0,h_0)\,dh,$$



Functions of more than 2 variables

§14.7 Extreme Values and Saddle Pts



$$\max_{x\in[a,b]}/\min f(x) = \max/\min \{f(a), f(b), f(c_i)\}$$

$f'(c_i)=0$ — c_i – critical pt

$\{a, b\}$ — boundary pts , $\{c_i\}$ — interior pts

Def. (1) $f(a,b)$ is a local maximum value of $f \Longleftrightarrow f(a,b) \geq f(x,y)$ $\forall (x,y) \in D_r(a,b)$

(2) " " " " minimum " " " $\Longleftrightarrow f(a,b) \leq f(x,y)$

Thrm (1^{st} Der. Test) Assume that

(1) f has a local max/min at an interior pt (a,b)

(2) f_x, f_y exist

$\Longrightarrow f_x(a,b)=0$ and $f_y(a,b)=0 \Longleftrightarrow \nabla f(a,b)=(0,0)$

Proof Assume that f has a local max at (a,b)

$\Rightarrow \forall (h_1,h_2) \in R^2$, $g(t) = f(a+th_1, b+th_2)$ has a local max at $t=0$

$\Longrightarrow 0 = g'(0) = \nabla f(a,b) \cdot (h_1,h_2)$

$\Longrightarrow \nabla f(a,b) = (0,0)$ #

Critical Pts (a,b) (1) (a,b) is an interior pt

(2) $\nabla f(a,b) = (0,0)$ or $\nabla f(a,b)$ DNE.

<u>Saddle Pts</u> (a,b) is a critical pt but not local max/min.

<u>Ex. 1</u> Find the local extreme values of $f(x,y) = x^2 + y^2 - 4y + 9$

<u>Ex. 2</u> $f(x,y) = y^2 - x^2$

<u>Thrm (2nd Der. Test)</u> Assume that

(1) $f, f_x, f_y, f_{xx}, f_{xy}, f_{yy} \in C^0(D_r(a,b))$ $= \{(x,y) \mid (x-a)^2 + (y-b)^2 < r^2\}$

(2) $\nabla f(a,b) = (0,0)$

$\Longrightarrow$ (1) $D = (f_{xx} f_{yy} - f_{xy}^2)(a,b) > 0$ $\begin{cases} f_{xx}(a,b) < 0 \Rightarrow f(a,b) \text{ is a l. max.} \\ f_{xx}(a,b) > 0 \Rightarrow f(a,b) \text{ is a l. min} \end{cases}$

discriminant or Hessian of f

(2) $D < 0 \Longrightarrow (a,b)$ is a saddle pt.

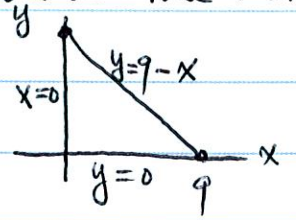
(3) $D = 0$ ~~thus~~ the test is inconclusive.

<u>Ex. 3</u> $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

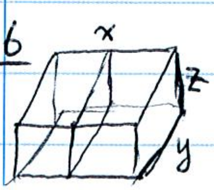
<u>Ex. 4</u> $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

<u>Ex. 5</u> Find the absolute max/min values of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$

on



<u>Ex. 6</u>



$\max V = xyz$ with $x + 2y + 2z = 108$

$= (108 - 2y - 2z)yz$

§14.8 Lagrange Multipliers

Constrained Maxima/Minima

Ex. 1 Find (x,y,z) on the plane $2x+\overset{g(x,y,z)}{\overset{\Vert}{y}}-z=5$ that is closest to $(0,0,0)$.

Solution

$$\min_{\substack{(x,y,z)\\ g(x,y,z)=c}} \sqrt{x^2+y^2+z^2} \iff \min_{\substack{(x,y,z)\\ g(x,y,z)=c}} \left(x^2+y^2+z^2\right)=f(x,y,z)$$

$$\iff \min f\big|_{g=c}$$

Ex. 0 $\min\left(x^2+y^2\right)$

$2x+y=5$

$$z=2x+y-5$$

$$h(x,y)=f\big|_{g=c}=x^2+y^2+\left(2x+y-5\right)^2$$

Ex. 2 Find (x,y,z) on $g(x,y,z)=x^2-z^2-1=0$ that are closest to $(0,0,0)$

Solution 1

$$\min_{g=0} f(x,y,z) \iff \min f\big|_{g=0}$$

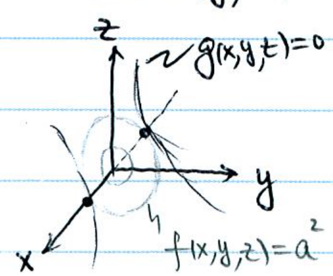
- $h(x,y)=f(x,y,z)\big|_{z^2=x^2-1}=x^2+y^2+\left(x^2-1\right)$

$\min h(x,y) \implies$ critical pt $(x,y)=(0,0) \Rightarrow z^2=0-1=-1$ wrong

- $k(y,z)=f(x,y,z)\big|_{x^2=z^2+1}=\left(z^2+1\right)+y^2+z^2 \geq 1$ — min at $(\pm 1,0,0)$

$\min k(y,z) \implies$ critical pt $(y,z)=(0,0) \Rightarrow x^2=1 \Rightarrow x=\pm 1$

Solution 2



$$\begin{cases}\nabla f=\lambda\nabla g\\ x^2-z^2-1=0\end{cases} \Rightarrow \begin{cases}2x=2\lambda x\\ 2y=0\\ 2z=2\lambda z\\ x^2-z^2-1=0\end{cases} \implies \overset{\lambda=1}{(\pm 1,0,0)}$$

The Method of Lagrange Multipliers

Assume that (1) $f(x,y,z)$ and $g(x,y,z)$ are diff.

(2) $\nabla g \neq 0$ when $g=0$

critical pts for $\underset{g=0}{\max/\min} f$ satisfies

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$

Ex. 3 $\quad \max/\min f(x,y) = xy$

$1 = g(x,y) = \frac{x^2}{8} + \frac{y^2}{2}$

· Geometry of the solution

Ex. 4 $\quad \max/\min f(x,y) = 3x + 4y$

$g(x,y) \equiv x^2 + y^2 = 1$

Lagrange Multipliers with 2 Constraints

$\max/\min f(x,y,z)$

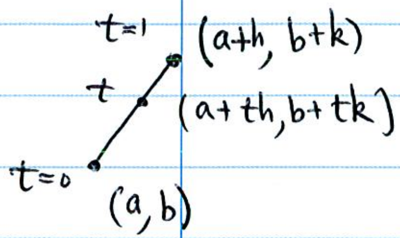
$g_1(x,y,z) = 0$

$g_2(x,y,z) = 0$

$$\begin{cases} \nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \\ g_1 = 0 \\ g_2 = 0 \end{cases}$$

Ex. 5

§14.9 Taylor's Formula for 2 Variables



$$F(t) = f(a+th,\, b+tk)$$

$$F'(t) = \nabla f \cdot (h,k) = hf_x + kf_y$$

$$F''(t) = (F')' = h\frac{\partial F'}{\partial x} + k\frac{\partial F'}{\partial y} = h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}$$

$$\Longrightarrow \quad F(1) = F(0) + F'(0)(1-0) + F''(c)\frac{(1-0)^2}{2}$$

$$= F(0) + F'(0) + \frac{1}{2}F''(c) \qquad c \in (0,1)$$

$$\Longleftrightarrow \quad f(a+h, b+k) = f(a,b) + hf_x(a,b) + kf_y(a,b) + \frac{1}{2}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)\Big|_{(a+ch,\, b+ck)}$$

diff. operator $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)$, $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 = h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x \partial y} + k^2\frac{\partial^2}{\partial y^2}$

$$\Longrightarrow \quad F^{(n)}(t) = \frac{d^n}{dt^n}F(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x,y)$$

$$F(t) = F(0) + F'(0)t + \frac{F''(0)}{2!}t^2 + \cdots + \frac{F^{(n)}(0)}{n!}t^n + \text{remainder}$$

$$\text{remainder} = \frac{F^{(n+1)}(c)}{(n+1)!}t^{n+1}$$

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \cdots + \frac{1}{n!}F^{(n)}(0) + \frac{1}{(n+1)!}F^{(n+1)}(c)$$

$$\Longrightarrow \quad f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f\Big|_{(a,b)} + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2\Big|_{(a,b)} + \cdots + \frac{1}{(n+1)!}(\quad)^{n+1}f\Big|_{(a,b)+c(h,k)}$$

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f\Big|_{(0,0)} + \cdots + \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f\Big|_{(0,0)} + \frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(cx,cy)}$$

Ex. Find a quadratic approx to $f(x,y) = \sin x \sin y$ near $(0,0)$. How accurate is the approx if $|x| \le 0.1$ and $|y| \le 0.1$?

2^{nd} Der. Test $\nabla f(a,b) = (0,0)$

$$f(a+h,\, b+k) - f(a,b) = \frac{1}{2}\Big(\underbrace{h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}}_{Q(0)}\Big)(a,b) + R_2$$

$$f_{xx}\, Q(0) = \left(h f_{xx} + k f_{xy}\right)^2 + \big(\underbrace{f_{xx} f_{xy} - f_{xy}^2}_{= D}\big) k^2$$

(1) $f_{xx} < 0$ and $D > 0 \implies f(a+h,\, b+k) - f(a,b) < 0$, l. min.

(2) $f_{xx} > 0$ and $D > 0 \implies$ 〃 > 0, l. max.

(3) $D < 0 \implies \exists\, h, k$ s.t. $f(a+h,\, b+k) - f(a,b) \begin{cases} > 0 \\ < 0 \end{cases}$, saddle pt

(4) $D = 0 \implies$ undetermined.

Error Formula for Linear Approximation

$$\left| f(x,y) - \cancel{f(x_0, y_0)} \right| \le \frac{1}{2} M \left(|x - x_0| + |y - y_0| \right)^2$$

$$Q(x,y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

$$E = f(x,y) - \ell(x,y) = \frac{1}{2}\Big[(x-x_0)^2 f_{xx} + 2(x-x_0)(y-y_0) f_{xy} + (y-y_0)^2 f_{yy} \Big]_{(x_0 + c(x-x_0),\; y_0 + c(y-y_0))}$$

$$\Rightarrow |E| \le \frac{1}{2}\left(|x-x_0|^2 |f_{xx}| + \cdots \right)$$

$$\le \frac{1}{2} M \left((x-x_0)^2 + 2|x-x_0||y-y_0| + (y-y_0)^2 \right) \qquad |f_{xx}| \le M,\ |f_{xy}| \le M,\ |f_{yy}| \le M$$

$$= \frac{1}{2} M \left(|x-x_0| + |y-y_0| \right)^2$$

§14.10 Partial Derivatives with Constrained Variable.

Ex. 1 $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$, $\frac{\partial w}{\partial x} = ?$

Case 1 (x, y) — indep. variable
(w, z) — dep. variable

$$w = x^2 + y^2 + (x^2+y^2)^2 \implies \frac{\partial w}{\partial x} = 2x + 4x^3 + 4xy^2$$

Case 2 x, z — indep. var.
w, y — dep. var.

$$w = x^2 + (z - x^2) + z^2 \implies \frac{\partial w}{\partial x} = 0$$

Ex. 2 $\frac{\partial w}{\partial x}(2,-1,1) = ?$ if $w = x^2 + y^2 + z^2$, $z^3 - xy + yz + y^3 = 1$
where x, y — indep. variable.

$$\frac{\partial w}{\partial x} = 2x + 2z\frac{\partial z}{\partial x}$$

$$3z^2\frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} = 0 \implies \frac{\partial z}{\partial x} = \frac{y}{y+3z^2} \implies \frac{\partial w}{\partial x} = ?$$

Notation $\left(\frac{\partial w}{\partial x}\right)_y$ — $\frac{\partial w}{\partial x}$ with x, y indep. var.

$\left(\frac{\partial w}{\partial y}\right)_{x,t}$ — $\frac{\partial w}{\partial y}$ with y, x, t indep. var.

Ex. 3 $\left(\frac{\partial w}{\partial x}\right)_{y,z}$ if $w = x^2 + y - z + \sin t$, $x + y = t$