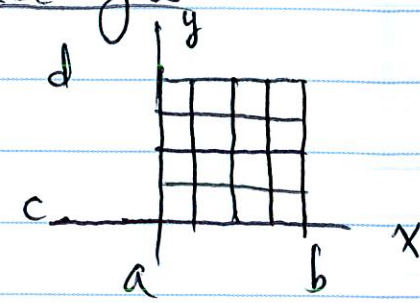
Chapter 15 Multiple Integrals

§15.1 Double and Iterated Integrals over Rectangles

$$R = [a,b] \times [c,d]$$



$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k$$

Fubini's Thrm

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

iterated integral

Ex. 1 $\iint_R f\,dA = ?$ $\quad f = 100 - 6x^2y$, $R = [0,2] \times [-1,1]$

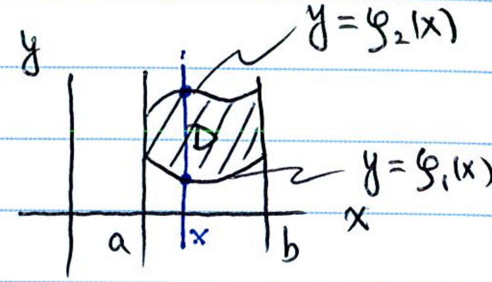
Ex. 2 $z = 10 + x^2 + 3y^2 = f(x,y)$ Volume $= \iint_R f(x,y)\,dA$

$R = [0,1] \times [0,2]$

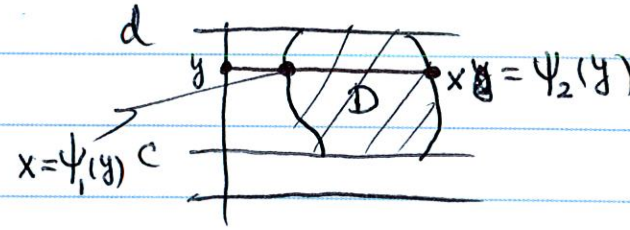
§15.2 Double Integrals over General Regions

Elementary Regions D

D is y-simple :
$$\begin{cases} a \le x \le b \\ \varphi_1(x) \le y \le \varphi_2(x) \end{cases}$$

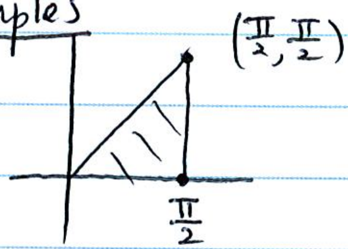


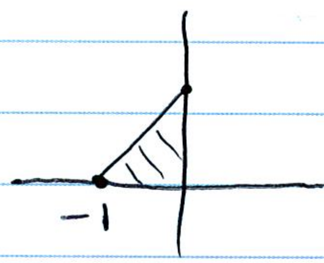
D is x-simple
$$\begin{cases} \psi_1(y) \le x \le \psi_2(y) \\ c \le y \le d \end{cases}$$

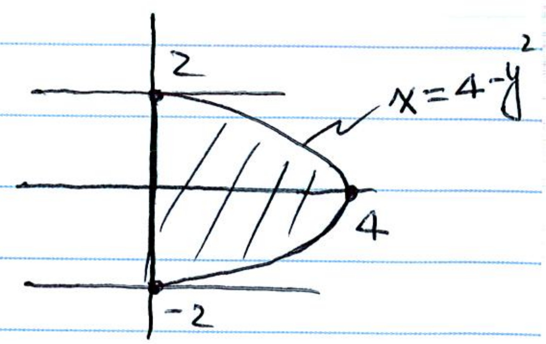


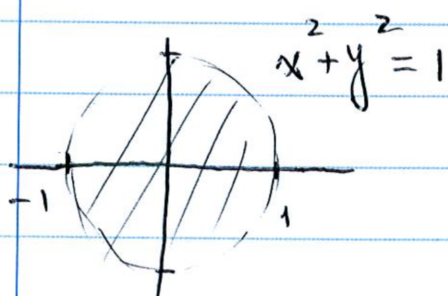
D is ~~elementary~~ simple region $\Longleftrightarrow$ D is either x-simple or y-simple

Examples









integral over elementary region D

$$\iint_D f\,dA = \iint_R f^*\,dA$$

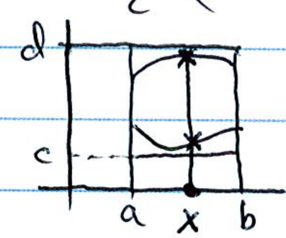
where $R \supset D$ is a rectangle, and $f^*(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \in R \setminus D \end{cases}$

- $\iint_D f\,dA$ is indep. of R (see iterated integral)

iterated integral $\quad R = [a,b] \times [c,d] \supset D$

$$\iint_D f\,dA = \iint_R f^*\,dA = \int_a^b \left(\int_c^d f^*(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f^*\,dx \right) dy$$

$$= \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f\,dy \right) dx \quad \text{if}$$



$$= \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy$$

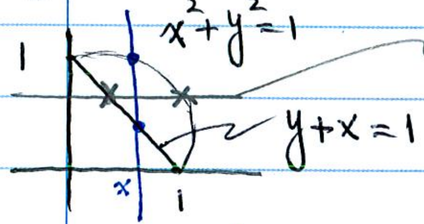
Ex. 1 Find the volume of the prism.

base: $y=0,\ y=x,\ x=1$

$z = f(x,y) = 3 - x - y$

Ex. 2 $\iint_R \frac{\sin x}{x}\,dA$, $\quad R$: the triangle by x-axis, $y=x$, $x=1$

Finding Limits of Integration



horizontal cross-section

$$\begin{cases} 1-y \le x \le \sqrt{1-y^2} \\ 0 \le y \le 1 \end{cases}$$

vertical cross-section

$$\begin{cases} 0 \le x \le 1 \\ 1-x \le y \le \sqrt{1-x^2} \end{cases}$$

Ex. 3 Sketch the region of integration $\displaystyle\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$

Properties of Double Integrals

(1) $\displaystyle\iint_D c\, f(x,y)\, dA = c \iint_D f\, dA$

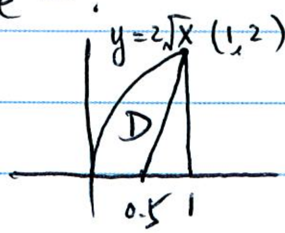
(2) $\displaystyle\iint_D (f \pm g)\, dA = \iint_D f\, dA \pm \iint_D g\, dA$

(3) $\displaystyle\iint_D f\, dA \ge \iint_D g\, dA$ if $f(x,y) \ge g(x,y)$ on D.

(4) $\displaystyle\iint_D f\, dA = \sum_{i=1}^{n} \iint_{D_i} f\, dA$ if $D = \bigcup_{i=1}^{n} D_i$ and $D_i \cap D_j = \phi$.

Ex. 4 Volume = ? $z = 16 - x^2 - y^2$.

D:



§15.3 Area by Double Integration

Area $|D| = \iint_D dA$ — area of a closed, bounded plane region D

Ex. 1 $A = \iint_D dA$ D: bounded by $y = x$, $y = x^2$ in the 1st quadrant

Ex. 2 $\iint_D dA$, D: $y = x^2$ and $y = x+2$.

Average Value

$$\text{average value of } f \text{ over } D = \frac{1}{\underbrace{|D|}_{\text{area of } D}} \iint_D f \, dA$$

Ex. 3 $f = x \cos xy$, $D = [0, \pi] \times [0, 1]$

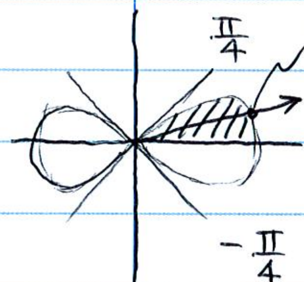
averages value of f over $D = ?$

§15.4 Double Integrals in Polar Form

area in polar coordinates

$$|D| = \iint_D r\,dr\,d\theta$$

Ex. 2 Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$



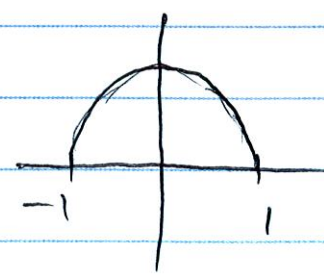
$$A = 4\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$$

$$\iint_D f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \operatorname{abs}\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \operatorname{abs}\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

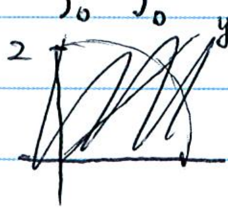
Jacobian

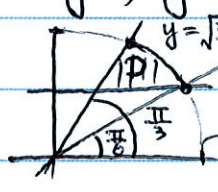
Ex. 3 $\iint_R e^{x^2+y^2}\,dx\,dy$, R: the semi-circular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$



Ex. 4 $\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$, Ex. 5 $\iint (9-x^2-y^2)\,dx\,dy$ $x^2+y^2 \le 1$

Ex. 6





$y = 1 \Leftrightarrow r\sin\theta = 1 \Rightarrow r = \csc\theta$

$|D| = ?$ $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_{\csc}^{2} r\,dr\,d\theta$

§15.5 Triple Integrals in Rectangular Coordinates

$$\iiint_{\Omega} F(x,y,z)\,dx\,dy\,dz = \lim \sum_{i,j,k} F(x_i, y_j, z_k)\,\Delta V_k$$

elementary regions

$$\begin{cases} (x,y) \in D \\ \varphi_1(x,y) \le z \le \varphi_2(x,y) \end{cases}, \quad \begin{cases} (x,z) \in D \\ \psi_1(x,z) \le y \le \psi_2(x,z) \end{cases}, \text{ or } \begin{cases} (y,z) \in D \\ h_1(y,z) \le x \le h_2(y,z) \end{cases}$$

Ex. 1 Find the volume of the region Ω enclosed by surfaces

$$z = x^2 + 3y^2$$
$$z = 8 - x^2 - y^2$$

$$x^2 + 3y^2 = 8 - x^2 - y^2 \Longrightarrow \boxed{x^2 + 2y^2 = 4} \quad D = \{(x,y) \mid x^2 + 2y^2 \le 4\}$$

Ex. 2 Set up limits of integration for evaluating $\iiint_D F\,dV$

D — tetrahedron with $(0,0,0)$, $(1,1,0)$, $(0,1,0)$, $(0,1,1)$

order — $dy\,dz\,dx$

Ex. 3 $F = 1$

average of $F = \dfrac{1}{|D|} \iiint_D F\,dV$

Ex. 4 $\dfrac{1}{|D|} \iiint_D xyz\,dV \qquad D: \begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le 2 \\ 0 \le z \le 2 \end{array}$

§15.6 Moments and Centers of Mass

Masses and 1st Moments | balance and torque of the object in a gravitational field.

δ — desity

2-variables

mass $M = \iiint_D \delta \, dV$ $\qquad$ $M = \iint_R \delta(x,y)\, dA$

first moment w.r.t. yz-plane $(x=0)$ $\qquad$ w.r.t. y-axis $(x=0)$

$$M_{yz} = \iiint_D x \, \delta(x,y,z)\, dV \qquad M_y = \iint_R x \delta \, dA$$

center of mass

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{M}\left(M_{yz}, M_{xz}, M_{xy}\right)$$

Ex. 1 $\delta = \text{const}$. $\;$ D: $(x,y) \in R = \{x^2+y^2 \le 4\}$

$$0 \le z \le 4 - x^2 - y^2.$$

Solution By symmetry, $\bar{x} = \bar{y} = 0$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{4}{3}$$

Ex. 2 $R: x^2 \le y \le x$ in the first quadrant.

~~center of mass~~ centroid of the region $= ?$

Moments of Inertia energy stored in a rotating shaft.

the moments of inertia about L

$$I_L = \iiint_D r^2 \delta(x,y,z)\, dV$$

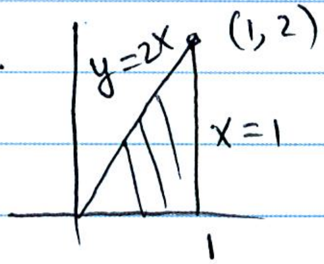
$r(x,y,z)$ — distance from (x,y,z) to L.

$L = x$-axis $\quad I_x = \iiint_D (y^2+z^2)\,\delta\, dV$

$I_y = \cdots;\quad I_z = \iiint_D (x^2+y^2)\,\delta\, dV$

Ex. 3 $I_x = ?,\ I_y = ?,\ I_z = ?$ $\quad \delta = \text{const.}$ $\quad D: \begin{array}{l} -\frac{a}{2} \le x \le \frac{a}{2} \\ -\frac{b}{2} \le y \le \frac{b}{2} \\ -\frac{c}{2} \le z \le \frac{c}{2} \end{array}$

Ex. 4



$\delta = 6x + 6y + 6$

$I_x = ?$

$I_y = ?$

§15.7 Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\iiint_D f(x,y,z)\,dV = \iiint_{D^*} f(r\cos\theta, r\sin\theta, z)\,dz\,r\,dr\,d\theta$$

Ex. 1 D: $z=0$, $x^2+(y-1)^2=1$, $z=x^2+y^2$

Ex. 2 $\delta=1$, D: $x^2+y^2=4$, $z=x^2+y^2$, xy-plane

centroid = ?

Spherical Coordinates

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases}$$

$$\iiint_D f(\rho,\varphi,\theta)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

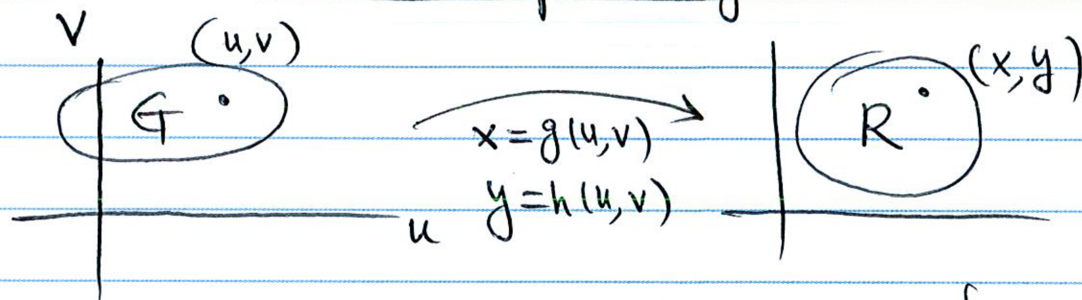
Ex. 3 spherical equation for $x^2+y^2+(z-1)^2=1$

Ex. 4 " " " $z=\sqrt{x^2+y^2}$

Ex. 5 $|D|=?$ $\rho\le 1$ and $\varphi=\frac{\pi}{3}$

Ex. 6 $I_z=?$ $\delta=1$ in Ex. 5

§15.8 Substitutions in Multiple Integral



$$\iint_R f(x,y)\,dx\,dy = \iint_G f\big(g(u,v),h(u,v)\big)\,\big|J(u,v)\big|\,du\,dv$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u} \quad \text{Jacobian}$$

Ex. 1 $\quad J = ? \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Ex. 2 $\quad \displaystyle\int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy = ? \quad \begin{cases} u = \frac{2x-y}{2} \\ v = \frac{y}{2} \end{cases}$,

Ex. 3 $\quad \displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$

$$\begin{cases} u = x+y \\ v = y-2x \end{cases} \Rightarrow \begin{cases} x = \frac{1}{3}(u-v) \\ y = \frac{1}{3}(2u+v) \end{cases}$$

Ex. 4 $\quad \displaystyle\int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\,dx\,dy \quad \begin{cases} u = \sqrt{xy} \\ v = \sqrt{\frac{y}{x}} \end{cases} \Rightarrow \begin{cases} x = \frac{u}{v} \\ y = uv \end{cases}$

$$\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases}$$

$$\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G H(u,v,z)\left|J(u,v,w)\right| du\,dv\,dw$$

$$H = F\big(g(u,v,w),\, h,\, k\big), \quad J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Longrightarrow |J| = r$$

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases} \qquad \left|J(\rho,\theta,\varphi)\right| = \rho^2\sin\varphi$$

Ex. 5 $$\int_0^3\int_0^4\int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$$

$$\begin{cases} u = \frac{1}{2}(2x-y) \\ v = \frac{y}{2} \\ w = z/3 \end{cases}$$