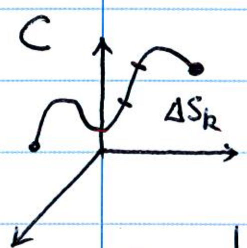
Chapter 16 Integration in Vector Fields

§16.1 Line Integrals

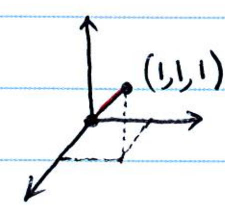
curve $C : \vec{r}(t) = \big(g(t), h(t), k(t)\big)$, $t \in [a,b]$

line integral $\displaystyle\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta S_k$

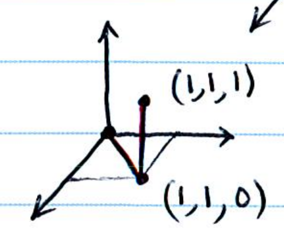


$$= \int_a^b f\big(\vec{r}(t)\big)\,\big|\vec{r}(t)\big|\,dt$$

Examples (1) $\displaystyle\int_{C_1} \left(x^2 - 3y^2 + z\right) ds$



(2) $\displaystyle\int_{C_2} \left(x - 3y^2 + z\right) ds$



mass $M = \displaystyle\int_C \delta\, ds$, δ – density

1st moments $M_{yz} = \displaystyle\int_C x\delta\, ds$, $M_{xz} = \displaystyle\int_C y\delta\, ds$, $M_{xy} = \displaystyle\int_C z\delta\, ds$

center of mass $\frac{1}{M}\left(M_{yz}, M_{xz}, M_{xy}\right)$

r – distance to L.

moments of inertia $I_x = \displaystyle\int_C (y^2+z^2)\delta\, ds$, $I_y = \displaystyle\int_C (x^2+z^2)\delta\, ds$, $I_L = \displaystyle\int_C r^2\delta\, ds$,

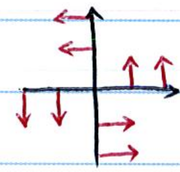
Ex. 3

§16.2 Vector Fields and Line Integrals: Work, Circulation, Flux

vector field $\vec{F}(x,y,z) = \left(M, N, P\right) : R^3 \to R^3$

$$\vec{F}(x,y) = \left(M(x,y), N(x,y)\right) : R^2 \to R^2$$

$\vec{F}(x,y) = (x, y)$



$\vec{F} = (-y, x) \big/ \sqrt{x^2+y^2}$

gradient field $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

line integrals of vector fields

unit tangent vector

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left(\vec{F} \cdot \frac{d\vec{r}}{ds}\right) ds = \int_a^b \left(\vec{F}\left(\vec{r}(t)\right) \cdot \frac{d\vec{r}}{dt}\right) dt$$

$$C : \vec{r}(t) = \left(g(t), h(t), k(t)\right), \ t \in [a,b]$$

$$= \int_C M\,dx + N\,dy + P\,dz$$

Examples (2) $\int_C \vec{F} \cdot d\vec{r}$, $\vec{F} = \left(z, xy, -y^2\right)$, $C : \vec{r}(t) = \left(t^2, t, \sqrt{t}\right)$, $t \in [0,1]$

(3) $\int_C -y\,dx + z\,dy + 2x\,dz$, $C : \vec{r}(t) = \left(\cos t, \sin t, t\right)$, $t \in [0, 2\pi]$

Work Done by a Force $\vec{F}$ – force

$$W = \int_C \vec{F} \cdot d\vec{r} \qquad \text{from } \vec{r}(a) \text{ to } \vec{r}(b)$$

Ex. 4, 5

Flow and Circulation for Velocity field $\vec{F}$ – velocity field of a fluid

$$\text{Flow} = \int_C \vec{F} \cdot d\vec{r} \qquad \text{from } \vec{r}(a) \text{ to } \vec{r}(b)$$

↳ circulation if $\vec{r}(a) = \vec{r}(b)$

Ex. 6, 7

Flux across a Simple Plane Curve $\vec{F} = (M, N)$

$C \subset \mathbb{R}^2$ is a simple curve $\Leftrightarrow$ it dose not cross itself

$C : \vec{r}(t) = (g(t), h(t)),\ t \in [a,b] \Leftrightarrow \forall\, t_1 \neq t_2 \in [a,b]$
$\Rightarrow \vec{r}(t_1) \neq \vec{r}(t_2)$

simple   not simple.

C – simple closed curve

outward normal

$$\text{Flux of } \vec{F} \text{ across } C = \int_C (\vec{F} \cdot \vec{n})\, ds = \oint_C M\,dy - N\,dx$$

$\vec{T} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right),\ \vec{n} = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$

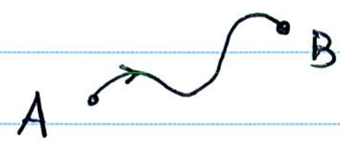
Ex. 8

§16.3 Path Independence, Conservative Fields, and Potential Functions

Fundamental Thrm of Line Integrals

$C: \vec{r}(t)$ $t\in[a,b]$ — a smooth curve joining pts from A to B

$$\Rightarrow \int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$



Proof $\displaystyle \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt$ #

Ex. 1 $\vec{F} = \nabla f$ — force field

$f = -\dfrac{1}{x^2+y^2+z^2}$, C joining $(1,0,0)$ to $(0,0,2)$

$? = \int_C \nabla f \cdot d\vec{r}$

Thrm The following statements are equivalent, $\vec{F} \in C^1(\mathbb{R}^3)^3$,

(1) $\forall$ simple closed curve C : $\int_C \vec{F}\cdot d\vec{r} = 0$

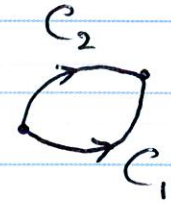
(2) $\vec{F}$ is conservative $\Leftrightarrow$ $\int_C \vec{F}\cdot d\vec{r}$ is path indep.

(3) $\vec{F} = \nabla f$

(4) $\nabla \times \vec{F} = \vec{0}$ on a simply connected domain.

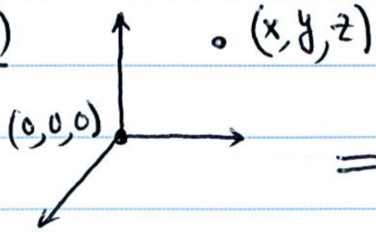
Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1)

$\underline{(1)\Rightarrow(2)}$



$$0=\int_{C_1\cup C_2^-}\vec{F}\cdot d\vec{r}=\int_{C_1}\vec{F}\cdot d\vec{r}-\int_{C_2}\vec{F}\cdot d\vec{r}$$

$\underline{(2)\Rightarrow(3)}$



if $\vec{F}=\nabla f$, then $f(x,y,z)-f(0,0,0)=\int_C\vec{F}\cdot d\vec{r}$

$$\Rightarrow f(x,y,z)=\int_C\vec{F}\cdot d\vec{r}$$

$C_1:(0,0,0)\to(x,0,0)\to(x,y,0)\to(x,y,z)$

$$\Rightarrow f(x,y,z)=\int_0^x M(t,0,0)\,dt+\int_0^y N(x,t,0)\,dt+\int_0^z P(x,y,t)\,dt$$

$$\Rightarrow \frac{\partial f}{\partial z}=P(x,y,z)$$

$\cdots$

$\underline{(3)\Rightarrow(4)}$ $\quad \nabla\times\nabla f=0$

$(4)\Rightarrow(1)$ $\quad \int_C\vec{F}\cdot d\vec{r}=\iint_S\nabla\times\vec{F}\cdot d\vec{S}=0$ #

$\underline{\text{Examples}}$ 3. Show that $\vec{F}=\left(e^x\cos y+yz,\ xz-e^x\sin y,\ xy+z\right)$ is conservative and find f s.t. $\nabla f=\vec{F}$

4. Show that $\vec{F}=\left(2x-3,\ -z,\ \cos z\right)$ is not conservative.

$\underline{\text{differential form}}$ $\quad M\,dx+N\,dy+P\,dz$

it is exact $\Leftrightarrow\ \exists f$ s.t. $M\,dx+N\,dy+P\,dz=df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$

$\Leftrightarrow\ \nabla\times(M,N,P)=0$

$\underline{\text{Ex. 6}}$

§16.4 Green's Thrm in the Plane

~~Divergence~~ $\vec{F} = (M, N) = M\vec{i} + N\vec{j}$ — vector field

Divergence $\quad \operatorname{div} \vec{F} = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}$

example 1

(a) $\vec{F} = (cx, cy)$ expansion/compression $\operatorname{div}\vec{F} = 2c$

(b) $\vec{F} = (-cy, cx)$ rotation $\operatorname{div}\vec{F} = 0$

(c) $\vec{F} = (y, 0)$ shearing flow $\operatorname{div}\vec{F} = 0$

Fig 16.28 (d) $\vec{F} = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$ whirlpool effect $\operatorname{div}\vec{F} = 0$

k-Component of Curl

$$\operatorname{curl}\vec{F}\cdot\vec{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \quad \text{— circulation density}$$

example 2

$\boxed{\begin{matrix}(a)\\ \vdots \\ (b)\end{matrix}}$ $\operatorname{curl}\vec{F}\cdot\vec{k} = \begin{cases} 0 & \text{no circulating} \\ 2c & \text{const circulating} \begin{cases} c>0 & \text{counter clockwise} \\ c<0 & \text{clockwise} \end{cases} \\ -1 & \\ 0 & \end{cases}$

Fig. 16.30 – 16.31

<u>Green's Thrm</u> R — a region in R^2, $\vec{F} = (M, N)$ — vector field

$C = \partial R$ — piecewise smooth, simple closed curve

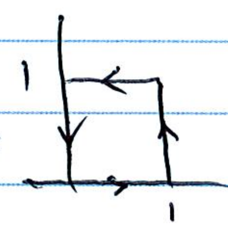
outward flux $\displaystyle\oint_C (\vec{F}\cdot\vec{n})\,ds = \oint_C M\,dy - N\,dx = \iint_R \operatorname{div}\vec{F}\,dx\,dy$.

counterclockwise circulation $\displaystyle\oint_C (\vec{F}\cdot\vec{T})\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\operatorname{curl}\vec{F}\cdot\vec{k}\right)dx\,dy$.

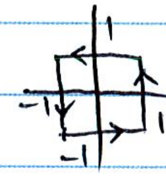
<u>Examples</u> 3. Verify the Green Thrm

$\vec{F} = (x-y, x)$, $C: \vec{r}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$

4. $\displaystyle\oint_C xy\,dy - y^2\,dx = ?$ C:



5. $\displaystyle\oint_C (\vec{F}\cdot\vec{n})\,ds = ?$ $\vec{F} = (x, y^2)$, C:



<u>Proof of Green's Thrm</u> (p957)

§16.5 Surfaces and Area

curves in R^2:
- graph $\quad y = f(x) \quad x \in [a,b]$
- level curve $\quad F(x,y) = 0$
- parametrized curve $\quad \vec{r}(t) = (f(t), g(t)),\ t \in [a,b]$

surfaces in R^3:
- graph $\quad z = f(x,y) \quad (x,y) \in D$
- level surface $\quad F(x,y,z) = 0$
- parametrized surface $\quad \vec{r}(u,v) = (f(u,v), g(u,v), h(u,v)),\ (u,v) \in D$

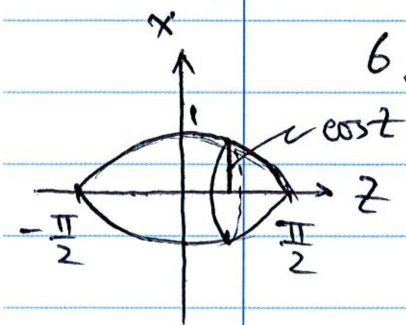
Examples ~~Compute~~ Find parametrization

1. $z = \sqrt{x^2+y^2},\ 0 \le z \le 1$
2. $x^2 + y^2 + z^2 = a^2$
3. $x^2 + (y-3)^2 = 9,\ 0 \le z \le 5.$

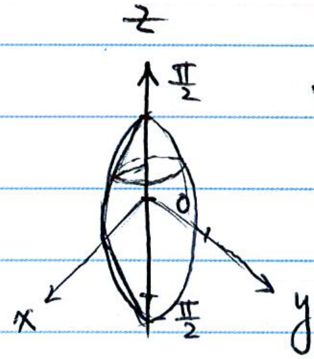
Surface $\quad S = \vec{r}(D)$

- S is smooth at $\vec{r}(u_0,v_0)$ $\iff$ $\vec{r}_u \times \vec{r}_v \big|_{(u_0,v_0)} \neq 0$
- $A(S) = \iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v|\, du\,dv$ $\quad$ surface area
- $dS = d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\,dv$ $\quad$ surface area differentials

Examples ~~4~~5. $A(S) = ?$ for examples 1 and 2.



6.



rotating the curve $x = \cos z, \; y = 0$, $-\frac{\pi}{2} \le z \le \frac{\pi}{2}$ around the z-axis.

parametrization $\vec{r}(u,v) = (\cos u \cos v, \; \cos u \sin v, \; u)$

$-\frac{\pi}{2} \le u \le \frac{\pi}{2}, \quad 0 \le v \le 2\pi$

$A(S) = ?$

Level surfaces $\quad F(x,y,z) = c, \quad (x,y) \in D$

Implicit Function Thrm $\Longrightarrow \quad z = h(u,v)$

parametrization : $\vec{r}(u,v) = (u, v, h(u,v))$

$\vec{r}_u = \left(1, 0, \frac{\partial h}{\partial u}\right) = \left(1, 0, -\frac{F_x}{F_z}\right)$

$\vec{r}_v = \left(0, 1, \frac{\partial h}{\partial v}\right) = \left(0, 1, -\frac{F_y}{F_z}\right)$

$F(u,v,h(u,v)) = 0$

$\Rightarrow \frac{\partial F}{\partial u} = F_x + F_z \cdot \frac{\partial h}{\partial u} = 0$

$\frac{\partial F}{\partial v} = F_y + F_z \frac{\partial h}{\partial v} = 0$

$\Downarrow$

$\frac{\partial h}{\partial u} = -\frac{F_x}{F_z}$

$\frac{\partial h}{\partial v} = -\frac{F_y}{F_z}$

$\Longrightarrow \vec{r}_u \times \vec{r}_v = \frac{1}{F_z}(F_x, F_y, F_z) = \frac{\nabla F}{F_z} = \frac{\nabla F}{\nabla F \cdot \vec{k}}$

$$\Longrightarrow \quad A(S) = \iint_D \frac{|\nabla F|}{|\nabla F \cdot \vec{k}|}\, dx\,dy$$

Ex. 7, 8

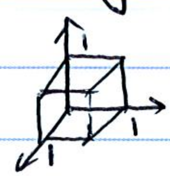
§16.6 Surface Integrals

$$\iint_S G(x,y,z)\, d\sigma = \begin{cases} \iint_D G(x,y,f(x,y)) \sqrt{1+f_x^2+f_y^2}\, dx\,dy \\ \iint_D G(x,y,z) \dfrac{|\nabla F|}{|\nabla F \cdot \vec{p}|}\, dA \\ \iint_D G(f(u,v), g(u,v), h(u,v)) \left|\vec{r}_u \times \vec{r}_v\right| du\,dv \end{cases}$$

Examples $\quad \iint_S G\, d\sigma = ?$

1. $G = x^2$ over $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$

2. $G = xyz$ over



3. $G = \sqrt{1-x^2-y^2}$ over $\vec{r}(u,v) = (\cos u \cos v,\ \cos u \sin v,\ u)$ $\quad u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $v \in [0, 2\pi]$.

Orientation

- S is oriented surface $\Longleftrightarrow$ S has two-sides (positive/negative)

Surface Integral for Flux $\quad \vec{F}$ — vector field in R^3

$$\text{Flux} = \iint_S (\vec{F} \cdot \vec{n})\, d\sigma$$

Examples $\iint_S \vec{F} \cdot \vec{n}\, d\sigma = ?$

4. $\vec{F} = \left(yz, x, -z^2\right)$ over $y = x^2$, $x \in [0,1]$, $z \in [0,4]$

5. $\vec{F} = \left(0, yz, z^2\right)$ over $\begin{cases} y^2 + z^2 = 1, \quad z \geq 0 \\ x = 0 \text{ and } x = 1 \end{cases}$

Moments and Masses of Thin Shells

§16.7 Stokes' Thrm

Curl $\quad \overrightarrow{\nabla \times F} = \operatorname{curl} \vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$

example $\quad \nabla \times \left(x^2 - z,\ x e^z,\ xy\right) = ?$

Stokes' Thrm

$$\oint_{C = \partial S} \vec{F} \cdot d\vec{r} = \iint_S \left(\overrightarrow{\nabla \times F}\right) \cdot \vec{n}\, d\sigma$$

Examples 2 and 3 $\quad \vec{F} = (y, -x, 0)$, $\quad S_1:$ $x^2 + y^2 + z^2 = 9,\ z \geq 0$ (hemi-sphere)

$C = \partial S:\ x^2 + y^2 = 9,\ z = 0$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{S_1} \left(\overrightarrow{\nabla \times F} \cdot \vec{n}\right) d\sigma$$

$S_2:$ $x^2 + y^2 \leq 9,\ z = 0$ (disc)

$$= \iint_{S_2} \left(\overrightarrow{\nabla \times F} \cdot \vec{n}\right) d\sigma$$

$C:\ x^2 + y^2 = 4,\ z = 2$

4. $\vec{F} = \left(x^2 - y,\ 4z,\ x^2\right)$

$$? = \oint_C \vec{F} \cdot d\vec{r} = \iint_{S_1} \left(\overrightarrow{\nabla \times F} \cdot \vec{n}\right) d\sigma$$

S_1: the cone

$$= \iint_{S_2} \left(\overrightarrow{\nabla \times F} \cdot \vec{n}\right) d\sigma$$

S_2: the disc

Paddle Wheel Interpretation of $\nabla\times\vec{F}$

the circulation density

see Fig. 16.60 $\quad \left(\nabla\times\vec{F}\right)\cdot\vec{u}\Big|_Q = \lim_{\rho\to 0}\frac{1}{\pi\rho^2}\oint_C \vec{F}\cdot d\vec{r}$

Examples 6 (p985) $\quad \vec{F} = \omega(-y, x)$, $\quad \omega$ – angular velocity of the rotation

$$\nabla\times\vec{F} = 2\omega\vec{k}$$

$$\nabla\times\vec{F}\cdot\vec{k} = \lim_{\rho\to 0}\frac{1}{\pi\rho^2}\oint_C \vec{F}\cdot d\vec{r} = 2\omega$$

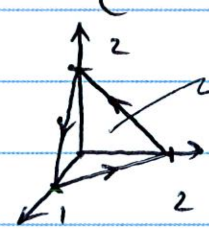
$C:\ x^2+y^2=\rho^2$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \left(\nabla\times\vec{F}\cdot\vec{n}\right)d\sigma = 2\omega\pi\rho^2$$

$S:\ x^2+y^2\le\rho^2$

7 (p985) $\quad \int_C xz\,dx + xy\,dy + 3xz\,dz = ?$

C:



8 (p986) $\quad S:\ z = x^2+4y^2$ and $z\le 1$.

orientation — positive $\vec{k}$-component.

$$\vec{F} = (y, -xz, xz^2)$$

$$? = \iint_S \nabla\times\vec{F}\cdot\vec{n}\,d\sigma$$

Identity $\quad \nabla\times\nabla f = 0 \quad$ and $\quad \nabla\times\nabla f = 0$

§16.8 The Divergence Thrm and a Unified Theory

Divergence $\quad \operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}, \quad \vec{F} = (M, N, P)$

example (a) expansion $\vec{F} = (x, y, z)$, $\operatorname{div}\vec{F} = 3$

(b) compression $\vec{F} = -(x, y, z)$, $\operatorname{div}\vec{F} = -3$

(c) rotation about z-axis: $\vec{F} = (-y, x)$, $\operatorname{div}\vec{F} = 0$

(d) shearing along horizantal planes; $\vec{F} = (0, z, 0)$, $\operatorname{div}\vec{F} = 0$

Divergence Thrm

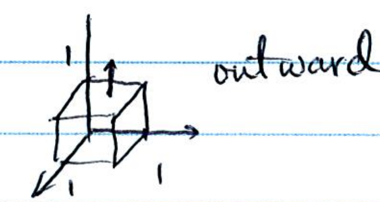
$$\iint_{S=\partial D} (\vec{F}\cdot\vec{n})\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,dV \qquad (2)$$

outward normal

Examples 2. $\vec{F} = (x, y, z)$, $S: x^2 + y^2 + z^2 = a^2$

verify (2).

3. $\vec{F} = (xy, yz, xz)$, $S =$



$$? = \iint_S (\vec{F}\cdot\vec{n})\,d\sigma$$

4. $\vec{F} = \frac{1}{\rho^3}(x, y, z)$, $\rho = \sqrt{x^2+y^2+z^2}$

$S = \partial D$ and $D : 0 < a^2 \le x^2 + y^2 + z^2 \le b^2$

$$? = \iint_S (\vec{F}\cdot\vec{n})\, d\sigma = \iiint_D \operatorname{div} \vec{F}\, dV = 0$$

S_a: $x^2 + y^2 + z^2 = a^2$

$$\iint_{S_a} (\vec{F}\cdot\vec{n})\, d\sigma = 4\pi .$$

Gauss' Law: (one of the 4 great laws of electromagnetic theory)

$\vec{E}(x,y,z) = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$, $\vec{r} = (x,y,z)$, q – point charge at (x,y,z)

electric field created by a pt charge q.

ε_0 — a physical const.

$\nabla\cdot\vec{E} = 0$, D — a region, and $\partial D \not\ni (0,0,0)$

$$\iint_{\partial D} (\vec{E}\cdot\vec{n})\, d\sigma = \begin{cases} \frac{q}{\varepsilon_0}, & (0,0,0) \in D \\ 0, & \text{otherwise} \end{cases}$$

Unifying the Integral Thrms (p998)