MATH 174 Quiz 8

Name: Solution PID: ____________

(2pts) 1. Find the spherical coordinates (ρ, ϕ, θ) and the cylindrical coordinates (r, θ, z) for the point $(x, y, z) = (1, 1, 1)$ in rectangular coordinates.

Sol. $\rho^2 = x^2+y^2+z^2 \Rightarrow \boxed{\rho = \sqrt{3}}$

$z = \rho\cos\phi \Rightarrow \cos\phi = \frac{z}{\rho} = \frac{1}{\sqrt{3}} \Rightarrow \boxed{\phi = \cos^{-1}\frac{1}{\sqrt{3}}}$

$\left.\begin{array}{l} x = \rho\sin\phi\cos\theta \\ \cos\phi = \frac{1}{\sqrt{3}} \Rightarrow \sin\phi = \frac{\sqrt{2}}{\sqrt{3}} \Rightarrow \rho\sin\phi = \sqrt{2} \end{array}\right\}$

$\Rightarrow \cos\theta = \frac{x}{\rho\sin\phi} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

$r^2 = x^2+y^2$

$\Rightarrow \boxed{r = \sqrt{2}}$

$\boxed{z = 1}$

Thus

$$\boxed{\begin{array}{l}(\rho, \phi, \theta) = (\sqrt{3}, \cos^{-1}\frac{1}{\sqrt{3}}, \frac{\pi}{4}) \\ (r, \theta, z) = (\sqrt{2}, \frac{\pi}{4}, 1)\end{array}}$$

(4pts) 2. Evaluate

$$\int_0^1 \int_{-2x}^{x} 2(x-y)\,dy\,dx$$

by applying the transformation

$$u = x - y, v = 2x + y$$

and integrating over an appropriate region in the uv-plane.

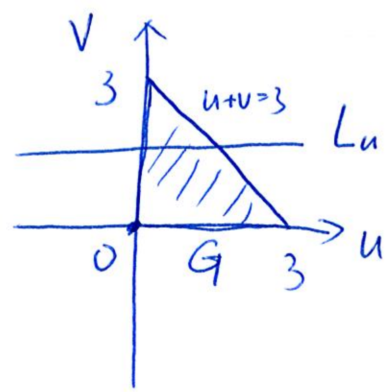
Sol. $\left.\begin{array}{l} u = x-y \\ v = 2x+y \end{array}\right\} \Rightarrow \begin{array}{l} x = \frac{u+v}{3} \\ y = \frac{-2u+v}{3} \end{array}$

$\Rightarrow f(x,y) = 2(x-y) = 2u$

$J(u,v) = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}$

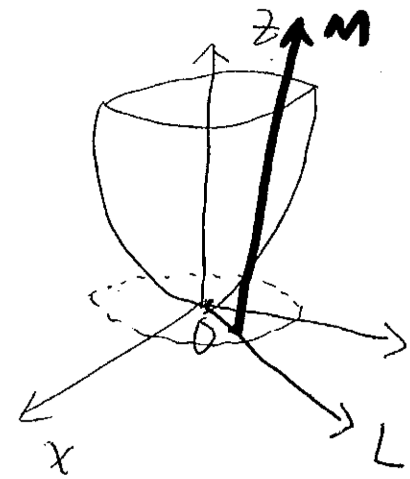
$$\boxed{\begin{aligned} &\int_0^1 \int_{-2x}^{x} 2(x-y)\,dy\,dx \\ &= \iint_G 2u \times \tfrac{1}{3}\,du\,dv \\ &= \int_0^3 \int_0^{3-v} \tfrac{2}{3}u\,du\,dv \\ &= 9 \end{aligned}}$$

Boundary of R	Boundary of G	Simplified equation
$y = x$	$\frac{-2u+v}{3} = \frac{u+v}{3}$	$u = 0$
$y = -2x$	$\frac{-2u+v}{3} = -2\,\frac{u+v}{3}$	$v = 0$
$x = 1$	$\frac{u+v}{3} = 1$	$u+v = 3$
$x = 0$	$\frac{u+v}{3} = 0$	$u+v = 0$



(4pts) 3. Find the center of mass of a solid of constant density bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$.

Sol: sketch the region D & its shadow R in xy-plane



By symmetry $\bar{x} = \bar{y} = 0$

only need to evaluate M & M_{xy}

use cylindrical coordinates

z-limits: M enters D at $z = x^2+y^2 = r^2$

M leaves D at $z = 4$

r-limits: Ray L $0 \le r \le 2$

θ-limits: $0 \le \theta \le 2\pi$

$$M = \iiint_D \delta \, dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 \delta \, dz \, r dr d\theta = 8\pi\delta$$

$$M_{xy} = \iiint_D \delta z \, dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 \delta z \, dz \, r dr d\theta = \frac{64}{3}\pi\delta$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{8}{3}$$

So the center of mass is $\boxed{(0, 0, \frac{8}{3})}$