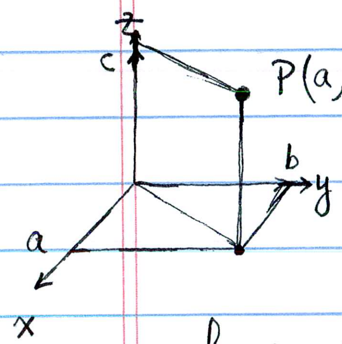
Calculus: Early Transcendentals, 7th Edition, J. Stewart

Chapter 12 Vectors and Geometry of Space

§12.1 Three-Dimensional Coordinate Systems



three-dimensional rectangular coordinate systems

surfaces (1) $z = 3$, (2) $y = 5$, (3) $x^2 + y^2 = 1$, (4) $y = x$

curves (1) $x^2 + y^2 = 1$ and $z = 3$

distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$: $|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

equation of a sphere with center $C(h, k, l)$ and radius r

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

examples (1) show that the equation represents a sphere, and find its center and radius

$$3x^2 + 3y^2 + 3z^2 = 10 + 6y + 12z$$ (#18, p790)

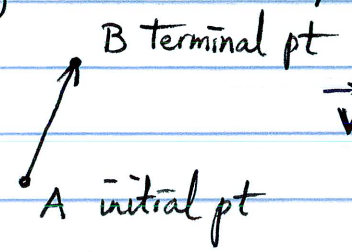
(2) write inequalities to describe the region

all pts between (but not on) the spheres of radius r and R centered at the origion, where $r < R$. (#37, p791)

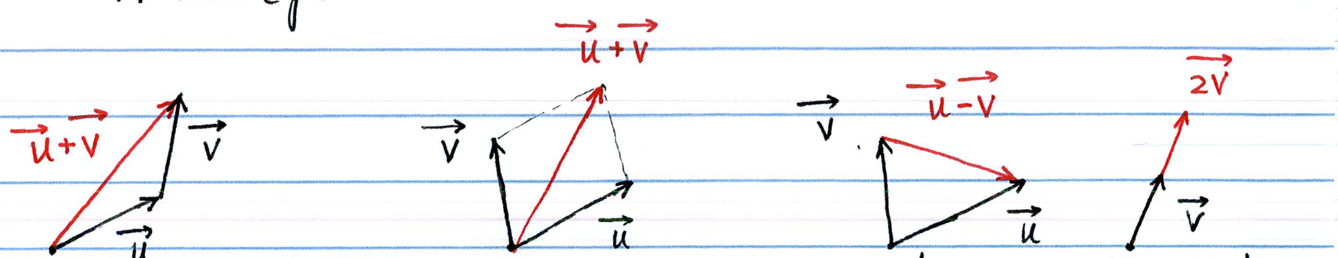
§12.2 Vectors

magnitude : mass, length, time, ...

vector (magnitude & direction) : force, displacement, velocity, ...

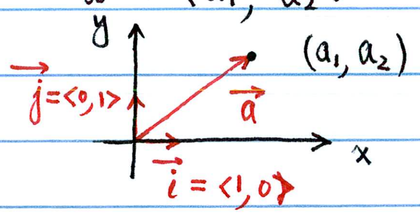


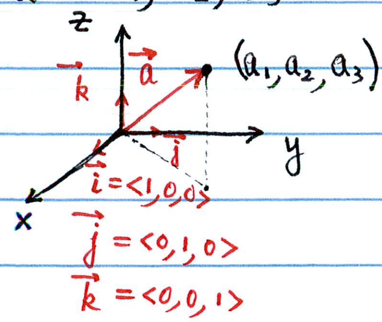
$\vec{v} = \vec{AB}$



adition subtraction scalar multiplication

components $\vec{a} = \langle a_1, a_2 \rangle$ $\vec{a} = \langle a_1, a_2, a_3 \rangle$





length or magnitude $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

adition and subtraction $\vec{a} \pm \vec{b} = \langle a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \rangle$

scalar multiplication $c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$

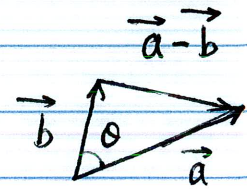
standard basis vectors $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$

unit vector $\dfrac{\vec{a}}{|\vec{a}|}$

§12.3 The Dot Product

$$\vec{a} = \langle a_1, a_2, a_3 \rangle, \quad \vec{b} = \langle b_1, b_2, b_3 \rangle$$

dot product $\quad \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$



Theorem $\quad \vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

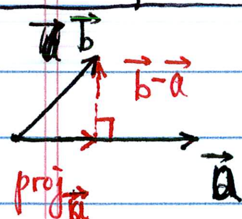
Proof

$$\left.\begin{array}{l} |\vec{a} - \vec{b}|^2 \underset{\text{the law of cosine}}{=} |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos\theta \\ \quad \parallel \\ |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b} \end{array}\right\} \Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta.$$

angel $\quad \cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$

orthogonality $\quad \vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$

projection of $\vec{b}$ onto $\vec{a}$ $\quad \mathrm{proj}_{\vec{a}} \vec{b} = \left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right) \dfrac{\vec{a}}{|\vec{a}|}$



$$\vec{b} - \vec{a} \perp \mathrm{proj}_{\vec{a}} \vec{b} = \alpha \vec{a} \implies \alpha = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}$$

examples (1) determine whether the given vectors are orthogonal, parallel, or neither

#24 (a) $\vec{u} = \langle -3, 9, 6 \rangle$, $\vec{v} = \langle 4, -12, -8 \rangle$

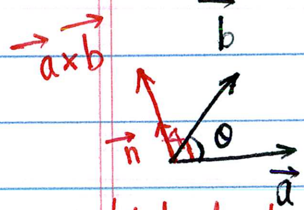
(b) $\vec{u} = \vec{i} - \vec{j} + 2\vec{k}$, $\vec{v} = 2\vec{i} - \vec{j} + \vec{k}$

(c) $\vec{u} = \langle a, b, c \rangle$, $\vec{v} = \langle -b, a, 0 \rangle$

#26 Find x such that the angel between $\langle 2, 1, -1 \rangle$ and $\langle 1, x, 0 \rangle$ is 45°.

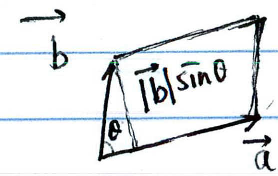
§12.4 The Cross Product

cross product $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\vec{k}$



$= \left(|\vec{a}|\,|\vec{b}|\sin\theta\right)\vec{n}$

right-hand rule



properties

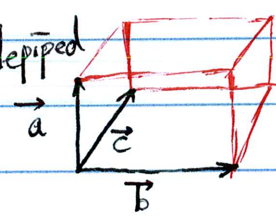
(1) $\vec{a}\times\vec{b} \perp \vec{a}$ and $\vec{a}\times\vec{b}\perp\vec{b}$

(2) $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$ = area of parallelogram

(3) $\vec{a}\times\vec{b} = \vec{0} \iff \vec{a}\,/\!/\,\vec{b}$

(4) $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$, $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$, $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

(5) $|\vec{a}\cdot(\vec{b}\times\vec{c})|$ = Volume of parallelepiped



examples (p814 – p815)

#4. $\vec{a} = \vec{j} + 7\vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + 4\vec{k}$; $\vec{a}\times\vec{b} = ?$, verify $\vec{a}\times\vec{b}\perp\vec{a}$ and $\vec{a}\times\vec{b}\perp\vec{b}$

#20. Find two unit vectors orthogonal to both $\vec{j}-\vec{k}$ and $\vec{i}+\vec{j}$

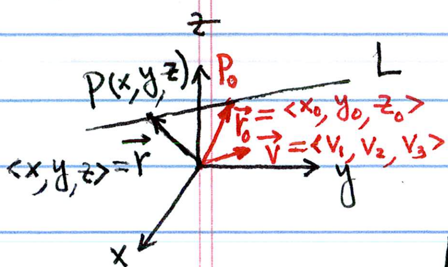
#28. Find the area of the parallelogram with vertices $(1,2,3)$, $(1,3,6)$, $(3,8,6)$, $(3,7,3)$

#29. $P(1,0,1)$, $Q(-2,1,3)$, $R(4,2,5)$
(a) Find a nonzero vector orthogonal to the plane
(b) Find the area of triangle PQR

§12.5 Equations of Lines and Planes

equation of a line Given $P_0(x_0, y_0, z_0)$ on line L

vector $\vec{v} \parallel L$



$P(x,y,z)$ — arbitrary pt on L

$$L: \quad \vec{r} = \vec{r}_0 + t\vec{v} \qquad t - \text{parameter}$$

parametric equation
$$\boxed{\begin{cases} x = x_0 + t v_1 \\ y = y_0 + t v_2 \\ z = z_0 + t v_3 \end{cases}}$$

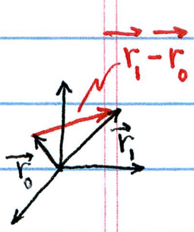
symmetric equation
$$\boxed{\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}}$$

examples (1) #3 (P824) Find a vector eq. and parametric eqs for line through the pt $(2, 2.4, 3.5)$ and parallel to $3\vec{i} + 2\vec{j} - \vec{k}$. also fine two other pts on the line

(2) #9 (P824) Find parametric eqs and sym. eqs for the line through the pts $(-8, 1, 4)$ and $(3, -2, 4)$. At what pt does this line intersect xy-plane?

$$\vec{v} = P_1 - P_0 = (-8, 1, 4) - (3, -2, 4) = (-11, 3, 0)$$

(3) #18 (P824) Find parametric eqs for the line segment from $\underset{\vec{r}_0}{(10, 3, 1)}$ to $\underset{\vec{r}_1}{(5, 6, -3)}$

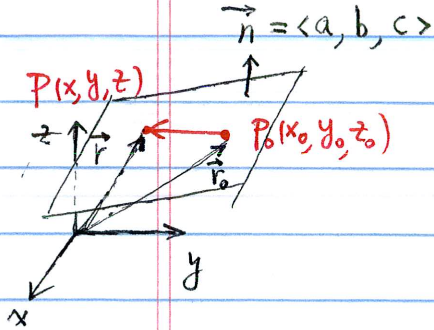


$$\boxed{\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \quad \text{for } t \in [0,1]}$$

$$\| $$

$$\vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$$

equation of a plane



Given pt $P_0(x_0, y_0, z_0)$ in the plane

vector $\vec{n} \perp$ plane

$$\boxed{0 = \vec{n} \cdot (\vec{r} - \vec{r}_0)}$$
$$= a(x-x_0) + b(y-y_0) + c(z-z_0)$$
$$\Longleftrightarrow \boxed{ax + by + cz = d}$$

examples (#28) Find an equation of the plane through the point $(2, 4, 6)$ parallel to the plane $z = x + y$

example 4 (P820) pt. $(2, 4, -1)$, normal $\vec{n} = \langle 2, 3, 4\rangle$

lesson 2

(#32) the plane through the origion and the pts $(2, -4, 6)$ and $(5, 1, 3)$.

(#15) (a) Find sym eqs for the line passes through the pt $(1, -5, 6)$ and ~~it~~ is parallel to the vector $\langle -1, 2, -3\rangle$

(b) Find the pts in which the required line in (a) intersects the coordinate planes

(example 6) line $\begin{cases} x = 2 + 3t \\ y = -4t \\ z = 5 + t \end{cases}$ plane $4x + 5y - 2z = 18$

(example 7) (a) find the angle between planes: $x + y + z = 1$ and $x - 2y + 3z = 1$

(b) find sym eq for the line of intersection of these two planes.

(a) $\vec{n}_1 = \langle 1, 1, 1\rangle$, $\vec{n}_2 = \langle 1, -2, 3\rangle \Longrightarrow \cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

(b) a pt = ? on xy-plane a vector = ? $\vec{n}_1 \times \vec{n}_2$

(#46) Find the pt at which the line intersects the given plane

$$\begin{cases} x = 1+2t \\ y = 4t \\ z = 2-3t \end{cases} \qquad x+2y-z+1=0$$

(# ~~58~~ 55) Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them

$$x = 4y-2z, \quad 8y = 1+2x+4z$$

§12.6 Cylinders and Quadric Surfaces

Cylinders : a surface that consists of all lines that parallel to a given line and pass through a given plane curve

Examples : Sketch (1) $z = x^2$

(2) $x^2+y^2=1$

(3) $y^2+z^2=1$

(#2) sketch (a) $y=e^x$ as a curve in R^2

(b) $y=e^x$ as a surface in R^3

Quadric Surfaces

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0$$

standard forms $\quad Ax^2+By^2+Cz^2+J=0$

or $\quad Ax^2+By^2+Iz=0$

example 3 use traces to sketch $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

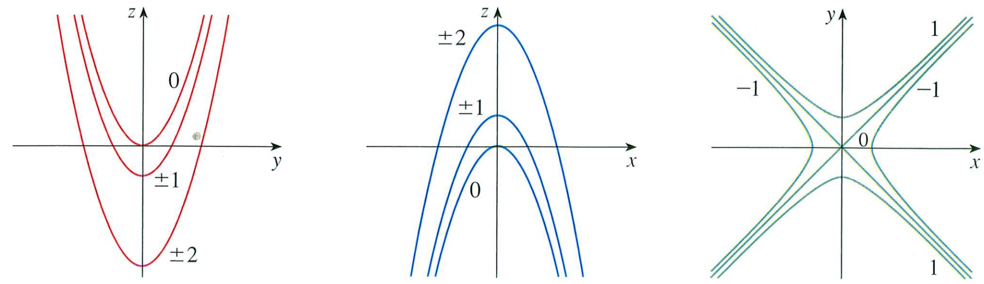
the curves of intersection
of the surface with planes
parallel to the coordinate planes

lessson 3

example 4 use traces to sketch the surface $z = 4x^2 + y^2$

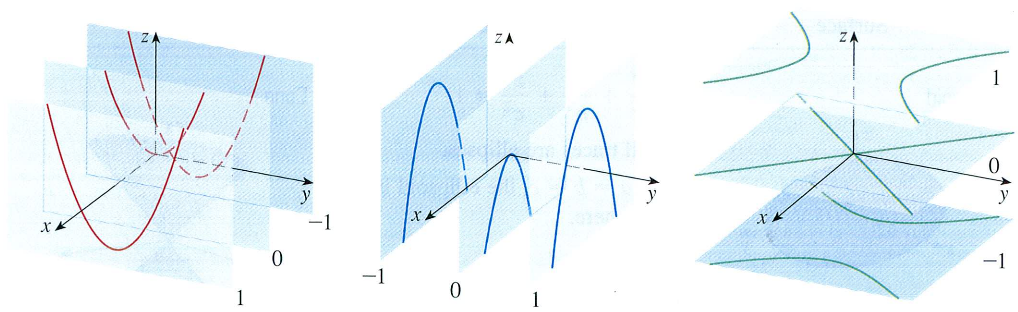
V EXAMPLE 5 Sketch the surface $z = y^2 - x^2$.

SOLUTION The traces in the vertical planes $x = k$ are the parabolas $z = y^2 - k^2$, which open upward. The traces in $y = k$ are the parabolas $z = -x^2 + k^2$, which open downward. The horizontal traces are $y^2 - x^2 = k$, a family of hyperbolas. We draw the families of traces in Figure 6, and we show how the traces appear when placed in their correct planes in Figure 7.



Traces in $x = k$ are $z = y^2 - k^2$ | Traces in $y = k$ are $z = -x^2 + k^2$ | Traces in $z = k$ are $y^2 - x^2 = k$

FIGURE 6
Vertical traces are parabolas; horizontal traces are hyperbolas. All traces are labeled with the value of k.



Traces in $x = k$ | Traces in $y = k$ | Traces in $z = k$

FIGURE 7
Traces moved to their correct planes

TEC In Module 12.6A you can investigate how traces determine the shape of a surface.

In Figure 8 we fit together the traces from Figure 7 to form the surface $z = y^2 - x^2$, a **hyperbolic paraboloid**. Notice that the shape of the surface near the origin resembles that of a saddle. This surface will be investigated further in Section 14.7 when we discuss saddle points.

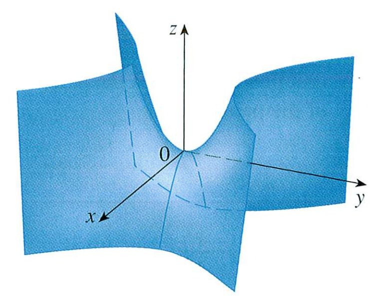


FIGURE 8
The surface $z = y^2 - x^2$ is a hyperbolic paraboloid.

EXAMPLE 6 Sketch the surface $\dfrac{x^2}{4} + y^2 - \dfrac{z^2}{4} = 1$.

SOLUTION The trace in any horizontal plane $z = k$ is the ellipse

$$\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4} \qquad z = k$$

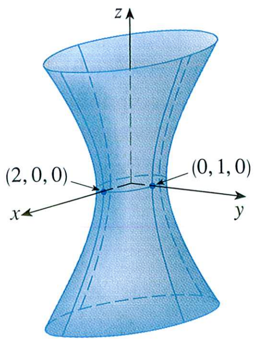


FIGURE 9

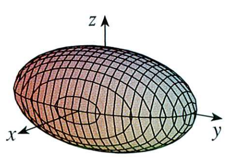
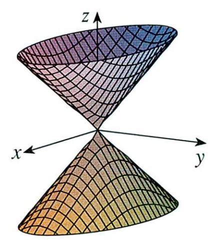
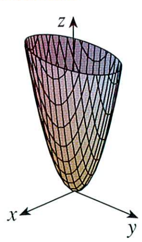
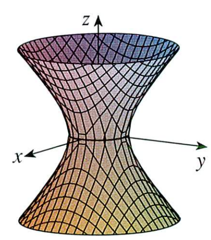
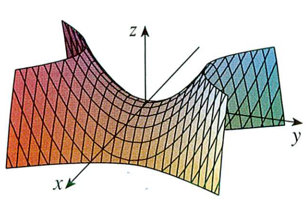
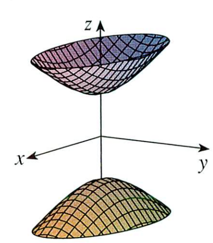
but the traces in the xz- and yz-planes are the hyperbolas

$$\frac{x^2}{4} - \frac{z^2}{4} = 1 \quad y = 0 \qquad \text{and} \qquad y^2 - \frac{z^2}{4} = 1 \quad x = 0$$

This surface is called a **hyperboloid of one sheet** and is sketched in Figure 9.

The idea of using traces to draw a surface is employed in three-dimensional graphing software for computers. In most such software, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k, and parts of the graph are eliminated using hidden line removal. Table 1 shows computer-drawn graphs of the six basic types of quadric surfaces in standard form. All surfaces are symmetric with respect to the z-axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.

TABLE 1 Graphs of quadric surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

TEC In Module 12.6B you can see how changing a, b, and c in Table 1 affects the shape of the quadric surface.

V EXAMPLE 7 Identify and sketch the surface $4x^2 - y^2 + 2z^2 + 4 = 0$.

SOLUTION Dividing by -4, we first put the equation in standard form:

$$-x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 1$$

Comparing this equation with Table 1, we see that it represents a hyperboloid of two sheets, the only difference being that in this case the axis of the hyperboloid is the y-axis. The traces in the xy- and yz-planes are the hyperbolas

$$-x^2 + \frac{y^2}{4} = 1 \qquad z = 0 \qquad \text{and} \qquad \frac{y^2}{4} - \frac{z^2}{2} = 1 \qquad x = 0$$

The surface has no trace in the xz-plane, but traces in the vertical planes $y = k$ for $|k| > 2$ are the ellipses

$$x^2 + \frac{z^2}{2} = \frac{k^2}{4} - 1 \qquad y = k$$

which can be written as

$$\frac{x^2}{\dfrac{k^2}{4} - 1} + \frac{z^2}{2\left(\dfrac{k^2}{4} - 1\right)} = 1 \qquad y = k$$

These traces are used to make the sketch in Figure 10.

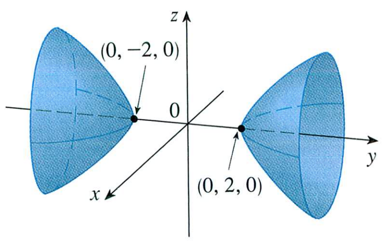


FIGURE 10
$4x^2 - y^2 + 2z^2 + 4 = 0$

EXAMPLE 8 Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

SOLUTION By completing the square we rewrite the equation as

$$y - 1 = (x - 3)^2 + 2z^2$$

Comparing this equation with Table 1, we see that it represents an elliptic paraboloid. Here, however, the axis of the paraboloid is parallel to the y-axis, and it has been shifted so that its vertex is the point $(3, 1, 0)$. The traces in the plane $y = k$ $(k > 1)$ are the ellipses

$$(x - 3)^2 + 2z^2 = k - 1 \qquad y = k$$

The trace in the xy-plane is the parabola with equation $y = 1 + (x - 3)^2$, $z = 0$. The paraboloid is sketched in Figure 11.

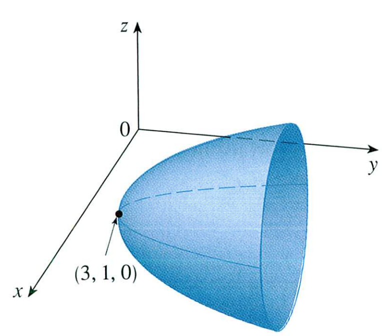


FIGURE 11
$x^2 + 2z^2 - 6x - y + 10 = 0$