Chapter 13 Vector Functions

Lessons
- 5. 13.1 (Beg.–Ex. 5)
- 6. 13.2
- 7. 13.3 (Beg. – Ex. 3)
- 8. 13.4 (Beg. – Ex. 5)

§13.1 Vector Functions and Space Curves

vector-valued functions $\vec{r}(t)$

3D rectangular coordinate $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

example (#2) Find the domain of $\vec{r}(t) = \langle \frac{t-2}{t+2}, \sin t, \ln(9-t^2) \rangle$

limit $\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$

example 2 $\lim_{t\to 0} \langle 1+t^3, te^{-t}, \frac{\sin t}{t} \rangle = ?$

continuity $\vec{r}$ is continuous at $t=a \iff \lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

space curves $C = \{ (f(t), g(t), h(t)) \mid t \in I \}$

parametric equations of C: $\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$ or $\langle x, y, z \rangle = \vec{r}(t)$

example 3 Describe the curve defined by $\vec{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle$

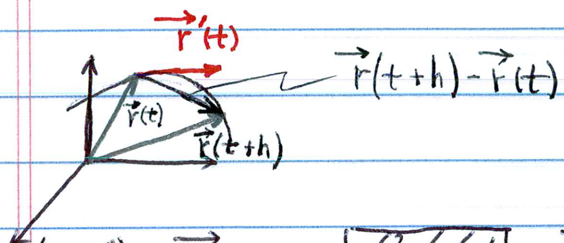
example 4 Sketch the curve whose vector eq. is $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

example 5. Find a vector eq. and parametric eq. for line segment joins $P(1,3,-2)$ to $Q(2,-1,3)$.

§13.2 Derivatives and Integrals of Vector Functions

Derivatives $\quad \dfrac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim\limits_{h\to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$

$$= \langle f'(t),\ g'(t),\ h'(t) \rangle$$



examples (#6) $\vec{r}(t) = \cancel{\langle t^2, t^3/3 \rangle}\ e^{t}\vec{i} + e^{-t}\vec{j}$,

(a) Sketch the plane curve; (b) Find $\vec{r}\,'(t)$;

(c) Sketch the position vector $\vec{r}(t)$ and the tangent vector $\vec{r}\,'(0)$

(#13) $\vec{r}(t) = \langle e^{t^2}, -1, \ln(1+3t) \rangle$, $\quad \vec{r}\,'(t) = ?$

(#24) $\begin{cases} x = e^t \\ y = te^t \\ z = te^{t^2} \end{cases}$ at $(1,0,0)$ $\quad$ Find the parametric eq of the tangent line.

Differentiation Rules $\quad \vec{u}, \vec{v}$ — differentiable; $c \in \mathbb{R}$, f — real-valued function

(1) $[\vec{u}(t) + \vec{v}(t)]' = \vec{u}\,'(t) + \vec{v}\,'(t)$; $\quad$ (2) $[c\,\vec{u}(t)]' = c\,\vec{u}\,'(t)$

(3) $[f(t)\,\vec{u}(t)]' = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}\,'(t)$;

(4) $[\vec{u}(t)\cdot\vec{v}(t)]' = \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$;

(5) $[\vec{u}(t)\times\vec{v}(t)]' = \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$; $\quad$ (6) $\dfrac{d}{dt}\vec{u}(f(t)) = \vec{u}\,'(f(t))\,f'(t)$.

example (#49) Find $f'(2)$, where $f(t) = \vec{u}(t) \cdot \vec{v}(t)$
$\vec{u}(2) = \langle 1, 2, -1 \rangle$, $\vec{u}'(2) = \langle 3, 0, 4 \rangle$, and $\vec{v}(t) = \langle t, t^2, t^3 \rangle$.

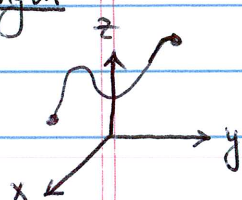
<u>Integrals</u>

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$$

example (#35) $\displaystyle\int_0^2 \langle t, -t^3, 3t^5 \rangle\,dt$

§13.3 Arc Length and Curvature

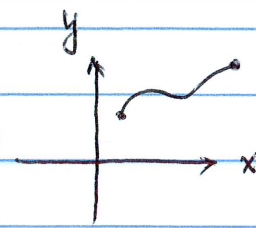
<u>arc length</u>



3D

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases} \quad a \le t \le b$$

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$
$a \le t \le b$



2D

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt$$

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$\boxed{L = \int_a^b |\vec{r}'(t)|\,dt} \quad \text{where } \vec{r}(t) = \langle f(t), g(t), h(t) \rangle.$$

<u>example</u> ~~4~~(#4, p860) $\vec{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle$, $0 \le t \le \frac{\pi}{4}$

$$\int \sec t\,dt = \int \frac{1}{\cos t}\,dt = \ln|\sec t + \tan t| + C$$

parametrizations of curve

C: unit circle $x^2+y^2=1$ $\begin{cases} \vec{r}_1=\langle \cos t, \sin t\rangle, & 0\le t\le 2\pi \\ \vec{r}_2=\langle \cos 2t, \sin 2t\rangle, & 0\le t\le \pi \end{cases}$

C: twisted cubic $\begin{cases} \vec{r}_1(t)=\langle t, t^2, t^3\rangle \\ \vec{r}_2(u)=\langle e^u, e^{2u}, e^{3u}\rangle \end{cases}$ $t=e^u$

arc length function $s=s(t)=\int_a^t |\vec{r}\,'(u)|\,du \Longrightarrow \frac{ds}{dt}=|\vec{r}\,'(t)|$

parametrization w.r.t. arc length s

$s=s(t)\Longrightarrow t=t(s)\Longrightarrow \vec{p}(s)=\vec{r}(t(s))$

example (#14) reparametrize $\vec{r}(t)=\langle e^{2t}\cos 2t, 2, e^{2t}\sin 2t\rangle$
from $t=0$ in the direction of increasing t

$s=s(t)=\sqrt{2}(e^{2t}-1) \Longrightarrow t=\frac{1}{2}\ln\left(1+\frac{s}{\sqrt{2}}\right)$

Curvature curve C is smooth $\Longleftrightarrow$ it has a smooth parametrization

$\vec{r}(t)$ is smooth $\Longleftrightarrow$ $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t)\ne 0$.

unit tangent vector $\vec{T}(t)=\dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

curvature of a curve $\kappa=\left|\dfrac{d\vec{T}}{ds}\right|$ measurement of how quickly the curve changes direction

example (#20) $\vec{r}(t)=\langle t, \frac{1}{2}t^2, t^2\rangle$ $=\left|\dfrac{d\vec{T}/dt}{ds/dt}\right|=\boxed{\dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}=\kappa}$

§13.4 Motion in Space: Velocity and Acceleration

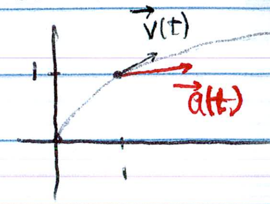
$\vec{r}(t)$ —— the position of a moving particle

$\vec{v}(t) = \vec{r}\,'(t)$ —— the velocity

$|\vec{v}(t)| = |\vec{r}\,'(t)| = \dfrac{ds}{dt}$ —— the speed

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$ —— the acceleration

example 1 position $\vec{r}(t) = \langle t^3, t^2 \rangle$. Find its velocity, speed, acceleration ~~at~~ at $t = 1$ and illustrate geometrically



example 3 Given $\vec{r}(0) = \langle 1, 0, 0 \rangle$, $\vec{v}(0) = \langle 1, -1, 1 \rangle$

$\vec{a}(t) = \langle 4t, 6t, 1 \rangle$

Find $\vec{v}(t) = ?$, $\vec{r}(t) = ?$

Newton's 2nd Law of Motion $\vec{F}(t) = m\,\vec{a}(t)$

example $\vec{r}(t) = \langle t^2, t, t^2 - t \rangle$. When is the speed a minimum?

$$s = |\vec{v}\,'(t)| = \sqrt{(2t)^2 + 1 + (2t-1)^2} = \sqrt{8\left(t - \tfrac{1}{4}\right)^2 + \tfrac{3}{2}}$$