Chapter 13 Vector Functions (4 lectures)

§13.1 Vector Functions and Space Curves

- <u>vector-valued functions</u>

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

- <u>Domain</u>

$$\mathrm{Dom}(\vec{r}(t)) = \{ t \mid f(t), g(t), h(t) \text{ are well defined} \}$$

$$= \mathrm{Dom}(f) \quad \mathrm{Dom}(g) \quad \mathrm{Dom}(h)$$

<u>Example (#2)</u> $\vec{r}(t) = \langle \frac{t-2}{t+2}, \sin t, \ln(9-t^2) \rangle$, $\mathrm{Dom}(\vec{r}) = ?$

- **limit**

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$$

- **continuity**

$\vec{r}(t)$ is continuous at a $\Longleftrightarrow$

examples

#3 $$\lim_{t\to 0} \langle e^{-3t}, \frac{t^2}{\sin^2 t}, \cos 2t \rangle$$

#5 $$\lim_{t\to \infty} \langle \frac{1+t^2}{1-t^2}, \tan^{-1} t, \frac{1-e^{-2t}}{t} \rangle$$

- representations of curves in 2D

(1) graphes

(2) level curves

(3) parametric curves

• <u>space curves</u>

$$C = \{ (f(t), g(t), h(t)) \mid t \in I \}$$

parametric equations of C $\quad \begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$ $\quad$ or $\quad \langle x, y, z \rangle = \vec{r}(t)$

<u>Ex. 3</u> Describe the curve defined by $\vec{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle$

<u>Ex. 4</u> Sketch the curve whose vector equation is $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

Ex. 5 Find a vector equation and parametric equations for the line segment that joins the points $(1,3,-2)$ and $(2,-1,3)$.

Ex. (#8) Sketch the curve of $\vec{r}(t) = \langle t^3, t^2 \rangle$. Indicate the direction in which t increases

Ex. (#30) At what points does the helix $\vec{r}(t) = \langle \sin t, \cos t, t \rangle$ intersect the sphere $x^2+y^2+z^2=5$?

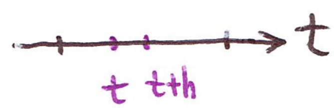
Ex. (#48) Two particles travel along the space curves

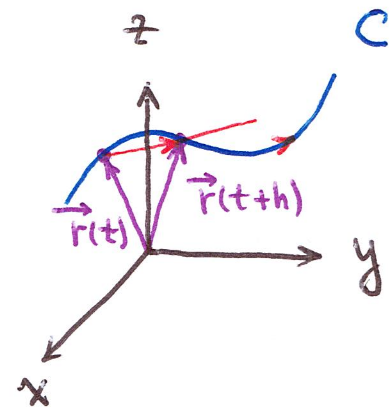
$$\vec{r}_1(t) = \langle t, t^2, t^3 \rangle, \quad \vec{r}_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle, \quad \text{for } t \geq 0$$

Do the particles collide? Do their paths intersect?

§13.2 Derivatives and Integrals of Vector Functions

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$





- derivative

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$=$$

- vector equation of tangent line at $\vec{r}(t_0)$

Ex. 1 Let $\vec{r}(t) = \langle 1+t^3, te^{-t}, \sin 2t \rangle$. (a) $\vec{r}'(t) = ?$ (b) Find the unit tangent vector at the point where $t=0$.

Ex. 2 Let $\vec{r}(t) = \langle \sqrt{t}, 2-t \rangle$, find $\vec{r}'(t)$ and sketch the position verctor $\vec{r}(1)$ and the tangent vector $\vec{r}'(1)$.

Ex. 3 Find parametric equations for the tangent line to the helix with para. eq. $\begin{cases} x = 2\cos t \\ y = \sin t \\ z = t \end{cases}$ at the point $(0, 1, \frac{\pi}{2})$.

Differentiation Rules

$\vec{u}(t), \vec{v}(t)$ — diff. vector functions, $f(t)$ — diff. scalar function

c — constant

(1) $\dfrac{d}{dt}\left[\vec{u}(t)+\vec{v}(t)\right] = \vec{u}'(t)+\vec{v}'(t)$; (2) $\dfrac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\vec{u}'(t)$

(3) $\dfrac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$

(4) $\dfrac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

(5) $\dfrac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

(6) $\dfrac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\vec{u}'(f(t))$

Proof of (4)

Ex. 4 Show that if $|\vec{r}(t)| = c$ (a constant), then $\vec{r}'(t) \perp \vec{r}(t)$ for all t.

Proof

Ex. (#49) Find $f'(2)$, where $f(t) = \vec{u}(t) \cdot \vec{v}(t)$

$\vec{u}(2) = \langle 1, 2, -1 \rangle$, $\vec{u}'(2) = \langle 3, 0, 4 \rangle$, and $\vec{v}(t) = \langle t, t^2, t^3 \rangle$.

- Integrals

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

example (#35)

$$\int_0^2 \langle t, -t^3, 3t^5 \rangle\,dt$$

§13.3 Arc Length and Curvature

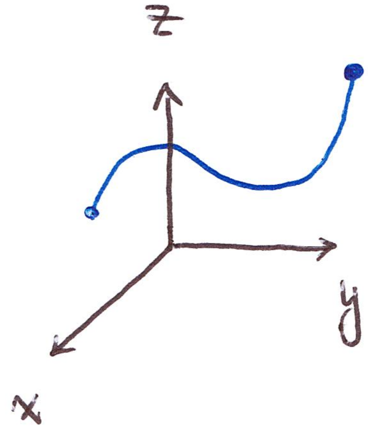
- arc length

$L =$

Curve : $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$t \in [a, b]$

traced exactly once



example (#4) $\vec{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle$, $t \in [0, \frac{\pi}{4}]$, $L = ?$

(hint : $\int \sec t \, dt = \int \frac{1}{\cos t} dt = \ln |\sec t + \tan t| + C$

- <u>parametrizations of a curve</u>

C : unit circle $x^2+y^2=1$ $\Bigg\langle$

- <u>arc length function</u>

$$s=s(t)=\int_a^t \left|\vec{r}\,'(u)\right| du \Longrightarrow \frac{ds}{dt}=\left|\vec{r}\,'(t)\right|$$

- <u>parametrization w.r.t. arc length s</u> $s=s(t) \Longrightarrow t=t(s)$

$$\vec{r}(t) \Longrightarrow \vec{p}(s)=\vec{r}(t(s))$$

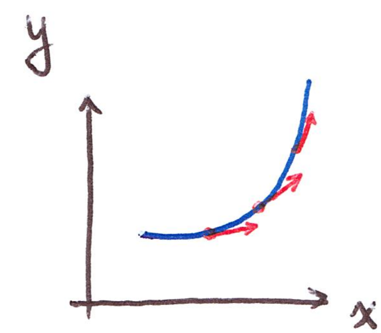
<u>Ex. (#14)</u> Reparametrize $\vec{r}(t)=\langle e^{2t}\cos(2t),\ 2,\ e^{2t}\sin(2t)\rangle$ w.r.t. arc length from $t=0$ in the direction of increasing t.

• Curvature (how a curve turns or bends)

Def. (1) $\vec{r}(t)$ is smooth $\Longleftrightarrow$

(2) C is a smooth curve $\Longleftrightarrow$

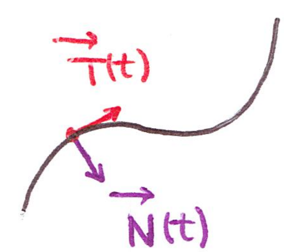
(3) the unit tangent vector $\vec{T}(t) =$



(4) curvature of a curve $\kappa =$

(measurement of how quickly the curve changes direction)

(5) principal unit normal vector $\vec{N}(t) =$



(6) binormal vector $\vec{B}(t) =$

examples

#19 $\vec{r}(t) = \langle e^{t}, t, e^{-t} \rangle$, find $\vec{T}$, $\vec{N}$, κ

<u>example</u> Find curvature of a circle with radius a.

§ 13.4 Motion in Space: Velocity and Acceleration

$\vec{r}(t)$ — the position of a moving particle

$\vec{v}(t) = \vec{r}\,'(t)$ — the velocity

$|\vec{v}(t)|$ — the speed $\qquad |\vec{v}(t)| = |\vec{r}\,'(t)| = \frac{ds}{dt}$

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$ — the acceleration

Ex. 1 The position vector of an object moving in a plane given by $\vec{r}(t) = \langle t^3, t^2 \rangle$. Find its velocity, speed, and acceleration when $t=1$ and illustrate geometrically.

Ex. 2 Find the velocity, acceleration, and speed of a particle with position vector $\vec{r}(t) = \langle t^2, e^t, te^t \rangle$.

Ex. 3 A moving particle starts at an initial position $\vec{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\vec{v}(0) = \langle 1, -1, 1 \rangle$. Its acceleration is $\vec{a}(t) = \langle 4t, 6t, 1 \rangle$. Find $\vec{v}(t)$ and $\vec{r}(t)$.

Ex. 4 An object with mass m that moves in a circular path with constant angular speed ω has position vector $\vec{r}(t) = \langle a\cos(\omega t), a\sin(\omega t)\rangle$.
Find the force acting on the object and show that it is directed toward the origin.

Newton's Second Law of Motion $\vec{F}(t) = m\,\vec{a}(t)$

Ex. 5 A projectile is fired with angle of elevation α and initial velocity $\vec{v}_0$. Assume that air resistance is negligible and the only external force is due to gravity. Find the position function $\vec{r}(t)$ of the projectile. What value of α maximizes the range (the horizontal distance traveled)?

$$\vec{F} = m\vec{a} = \langle 0, -mg \rangle, \qquad g = 9.8 \text{ m/s}^2$$