Chapter 14 Partial Derivatives (10 lessons)

(Lesson 1) §14.1 Functions of Several Variables

- functions of two variables

$z = f(x, y)$

dependent variable: z; indep. variables: x, y

domain $D = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \text{ is well-defined}\}$

range $R = \{f(x, y) \mid (x, y) \in D\}$

$\boxed{\forall (x, y) \in D,\ f(x, y) \text{ has a single value}}$

example Find and sketch the domain of the function

#13 $f(x, y) = \sqrt{2x - y}$, #19 $f(x, y) = \dfrac{\sqrt{y - x^2}}{1 - x^2}$

- graphs of $z = f(x, y)$ $\{(x, y, f(x, y)) \mid (x, y) \in D\}$

example Sketch the graph of (1) $f(x, y) = 6 - 3x - 2y$, (2) $g(x, y) = \sqrt{9 - x^2 - y^2}$

- level curves $\{(x, y) \mid f(x, y) = k\}$ k — constant

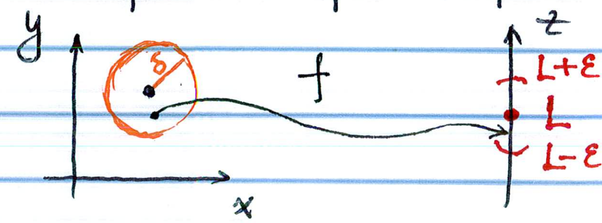
example sketch some level curves of (1) $f(x, y) = 6 - 3x - 2y$, (2) $h(x, y) = 4x^2 + y^2 + 1$

- functions of three variables $w = f(x, y, z)$ $\begin{cases} \text{domain} \\ \text{graph} \\ \text{level surface} \end{cases}$

example $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$

(Lesson 2) §14.2 Limits and Continuity

• Limits $\lim_{(x_0,y_0)\to(x,y)} f(x,y) = L \iff$ $\forall \varepsilon>0, \exists \delta>0$, s.t. "$0<|(x,y)-(x_0,y_0)|<\delta \Rightarrow |f(x,y)-L|<\varepsilon$"



• $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ DNE $\iff \lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{along path } C_1}} f(x,y) = L_1 \neq L_2 = \lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{along path } C_2}} f(x,y)$

examples (1) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ DNE.

Do the following limits exist?

(2) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$, (3) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^4+y^4}$

??(4) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = ?$ (5) $\begin{cases} \lim_{(x,y)\to(a,b)} x = a, \quad \lim_{(x,y)\to(a,b)} c = c \\ \lim_{(x,y)\to(a,b)} y = b, \end{cases}$

• Continuity $f(x,y)$ is continuous at $(a,b) \iff \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

examples (5) $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = ?$

(6) Where is $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ continuous?

(7) Is $g(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$ continuous? (8) p898

(Lesson 3) §14.3 Partial Derivatives

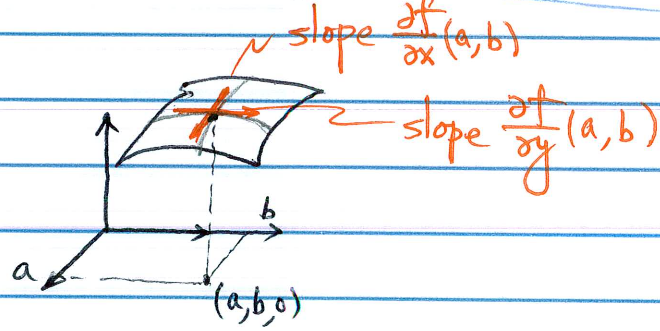
(beg. – Ex. 7)

$$z = f(x, y)$$

Definition $f_x(a,b) = \lim\limits_{h \to 0} \dfrac{f(a+h,b) - f(a,b)}{h}$, $f_y(a,b) = \cdots$

notations $f_x(a,b) = \dfrac{\partial f}{\partial x} = \dfrac{\partial z}{\partial x} = f_x$

Geometric Interpretation



examples (1) $f = x^2 + 3xy + y - 1$, $\dfrac{\partial f}{\partial x}(4,-5) = ?$, $\dfrac{\partial f}{\partial y}(4,-5) = ?$

(2) $f = y\sin(xy)$, $\dfrac{\partial f}{\partial y} = ?$

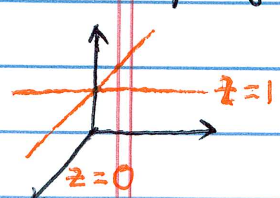
(3) $f = \dfrac{2y}{y + \cos x}$, $f_x = ?$, $f_y = ?$

(4) $yz - \ln z = x + y$, $\dfrac{\partial z}{\partial x} = ?$

Partial Derivative and Continuity (section 14.4)

$$f(x,y) = \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$

- $\lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$
- f is not cont. at $(0,0)$
- $\dfrac{\partial f}{\partial x}(0,0) = ?$ $\dfrac{\partial f}{\partial y}(0,0) = ?$



Higher-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^3 f}{\partial x \partial y^2}, \quad \cdots$$

example $f = 1 - 2xy^2z + x^2y$, $\dfrac{\partial f(x,y,z)}{\partial y \partial x \partial y \partial z} = ?$

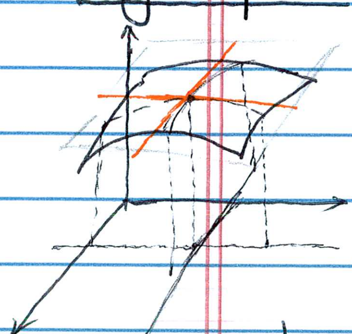
Clairaut's Thrm f is defined on a disk $D \ni (a,b)$.

$$f_{xy} \text{ and } f_{yx} \text{ are cont. on } D \implies f_{xy}(a,b) = f_{yx}(a,b)$$

§14.4 Tangent Planes and Linear Approximations

(Lesson 4, no Ex. 5, 6)

• tangent planes



surface $S: z = f(x,y)$

plane tangent to S at $(x_0, y_0, z_0 = f(x_0,y_0))$:

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

examples #1 $z = 3y^2 - 2x^2 + x$ at $(2, -1, -3)$

#4 $z = xe^{xy}$ at $(2, 0, 2)$

• Linear Approximations

$$f(x,y) \approx \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}_{\text{linear approximation of } f(x,y)}$$

example $f(x,y) = 2x^2 + y^2$ at $(1,1,3)$

$$\approx 3 + 4(x-1) + 2(y-1)$$

$$\Rightarrow f(1.1, 0.95) \approx 3 + 4(1.1-1) + 2(0.95-1) = 3.3$$

• Partial Derivative and Continuity

$$f(x,y) = \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$

- $f(x,y)$ is not cont. at $(0,0)$
- $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$

• Differentiability

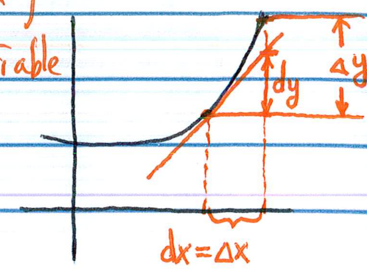
$f(x)$ is diff. at $x_0 \iff \lim_{x \to x_0} \frac{f(x) - [f(x_0) + f'(x_0)(x-x_0)]}{x - x_0} = 0$

$f(x,y)$ is diff at $(x_0,y_0) \iff \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - [f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)]}{|(x,y) - (x_0,y_0)|} = 0$

Thrm f_x and f_y are cont. at an open region containing (x_0,y_0)

$$\Longrightarrow f \text{ is diff at } (x_0,y_0).$$

- **Differentials** (linear approx to the change in function)



$y = f(x)$ $\quad \Delta y = f(x + \Delta x) - f(x) \approx f'(x)\,\Delta x$

the differential of y $\quad dy = f'(x)\,\underline{dx}$ $\quad$ indep. variable, the differential of x

$\underline{z = f(x, y)}$

the differential in z $\quad \boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$

example (#32) $\quad z = x^2 - xy + 3y^2$

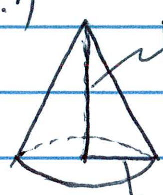
(a) $dz = ?$

(b) (x, y) changes from $(2, 3)$ to $(2.05, 2.96)$, compare Δz and dz

$dx = \Delta x = 0.05, \quad dy = \Delta y = -0.04$

(Lesson 5, §14.4 Ex. 5&6, §14.5 beg. – Ex. 7)

example 5



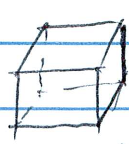
Use differentials to estimate the max. error in the calculation of the volume.

$V = \frac{1}{3}\pi r^2 h$

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \frac{2\pi r h}{3}dr + \frac{\pi r^2}{3}dh = \frac{\pi}{3}\left[100 \times 0.1 + 500 \times 0.1\right]$$

$\underline{w = f(x, y, z)}$ $\quad \boxed{dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz}$

example 6



$V = xyz, \quad x = 75, \; y = 60, \; z = 40, \; |\Delta x| \le 0.2$, $|\Delta z|, |\Delta y| \le 0.2$

$? = dV$

§14.5 Chain Rule

$$y = f(x),\ x = g(t) \Longrightarrow y = f(g(t))$$

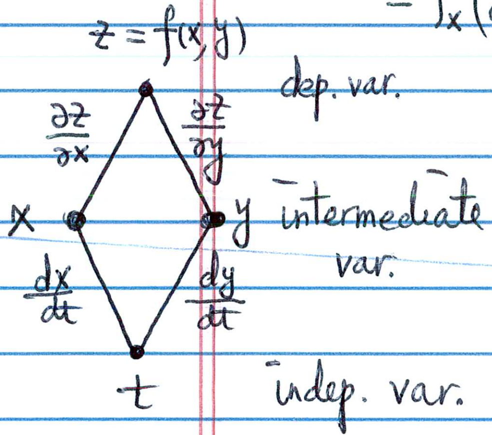
$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = f'(g(t))\,g'(t)$$

Functions of two variables

• $z = f(x,y)$, $\begin{cases} x = g(t) \\ y = h(t) \end{cases} \Longrightarrow z = f\big(g(t), h(t)\big)$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= f_x\big(g(t), h(t)\big)\,g'(t) + f_y\big(g(t), h(t)\big)\,h'(t)$$



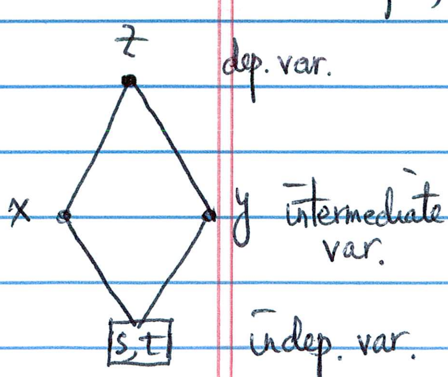
#2 $z = \cos(x + 4y)$, $\begin{cases} x = 5t^4 \\ y = \frac{1}{t} \end{cases}$, $\frac{dz}{dt} = ?$

example 2 Given $\underset{\text{pressure}}{P}\,\underset{\text{volume}}{V} = 8.31\,\underset{\text{temperature}}{T}$

at time t $T = 300$, $\frac{dT}{dt} = 0.1$, $V = 100$, $\frac{dV}{dt} = 0.2$

Find $\frac{dP}{dt} = ?$

• $z = f(x,y)$, $\begin{cases} x = g(s,t) \\ y = h(s,t) \end{cases} \Longrightarrow z = f\big(g(s,t), h(s,t)\big)$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

examples $z = x^2 + y^2$, $\begin{cases} x = s - t \\ y = s + t \end{cases}$, $\frac{\partial z}{\partial s} = ?$ $\frac{\partial z}{\partial t} = ?$

#11 $z = e^r \cos\theta$, $\begin{cases} r = st \\ \theta = \sqrt{s^2 + t^2} \end{cases}$, $\frac{\partial z}{\partial s} = ?$ $\frac{\partial z}{\partial t} = ?$

- $u = f(x_1, \cdots, x_n)$ and $\begin{cases} x_1 = x_1(t_1, \cdots, t_m) \\ \vdots \\ x_n = x_n(t_1, \cdots, t_m) \end{cases}$ $\Rightarrow u = f\left(x_1(t_1, \cdots, t_m), \cdots, x_n(t_1, \cdots, t_m)\right)$

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

examples 4. $w = f(x, y, z, t)$, $\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \\ t = t(u,v) \end{cases}$ $\frac{\partial w}{\partial u} = ?$ $\frac{\partial w}{\partial v} = ?$

#23 $w = xy + yz + zx$ $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r\theta \end{cases}$

$\frac{\partial w}{\partial r} = ?$ $\frac{\partial w}{\partial \theta} = ?$ at $r = 2$, $\theta = \frac{\pi}{2}$

6. $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ show that $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$

7. $z = f(x, y)$, $\begin{cases} x = r^2 + s^2 \\ y = 2rs \end{cases}$, $\frac{\partial z}{\partial r} = ?$, $\frac{\partial^2 z}{\partial r^2} = ?$

(Lesson 6, §14.5 implicit diff
§14.6 beg. – Ex. 5)

Implicit Differentiation

$$F(x,y)=0 \overset{\text{implicit}}{\Longrightarrow} y=f(x) \Longrightarrow F(x,f(x))=0 \Longrightarrow \frac{d}{dx}\Big(F(x,y(x))=0\Big)$$

$$0=\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx} \Longrightarrow \frac{dy}{dx}=-\frac{F_x}{F_y}$$

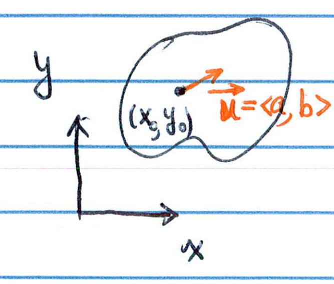
examples #28 $\quad \cos(xy)=1+\sin y, \quad \frac{dy}{dx}=?$

$$F(x,y,\underset{\substack{\parallel \\ f(x,y)}}{z})=0 \Longrightarrow \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

example #31 $\quad x^2+2y^2+3z^2=1, \quad \frac{\partial z}{\partial x}=?, \quad \frac{\partial z}{\partial y}=?$

§14.6 Directional Derivative and the Gradient Vector

$$z=f(x,y), \quad (x,y)\in D\subset\mathbb{R}^2$$

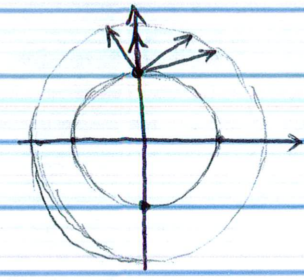


Question: In which direction at (x_0,y_0) does the value of $f(x,y)$ increase most rapidly?

$\Longleftrightarrow$ Find $\vec{u}=\langle a,b\rangle$ s.t.

$$\frac{d}{dt}f\big((x_0,y_0)+t\vec{u}\big)\Big|_{t=0}=\max_{\substack{\vec{u}\in\mathbb{R}^2\\|\vec{u}|=1}}\frac{d}{dt}f\big((x_0,y_0)+t\vec{u}\big)\Big|_{t=0}$$

example $f(x,y) = x^2+y^2$



$\vec{u} = \langle a, b\rangle$

$$\max_{\substack{\vec{u}\in R^2 \\ |\vec{u}|=1}} \frac{d}{dt} f\left((x_0,y_0)+t\vec{u}\right)\Big|_{t=0}$$

$$= \max_{|\vec{u}|=1} \left\{ \frac{\partial f}{\partial x}(x_0,y_0)\, a + \frac{\partial f}{\partial y}(x_0,y_0)\, b \right\}$$

$$\boxed{\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \quad \text{gradient vector}}$$

$$= \max_{|\vec{u}|=1} \nabla f(x_0,y_0)\cdot \vec{u}$$

$$= \max_{|\vec{u}|=1} \left|\nabla f(x_0,y_0)\right| \cos\theta$$

$$= \left|\nabla f(x_0,y_0)\right| \quad \text{if } \theta = 0 \iff \vec{u} \parallel \nabla f(x_0,y_0)$$

directional derivative

$$D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\vec{u} = \lim_{h\to 0} \frac{f\left((x_0,y_0)+h\vec{u}\right) - f(x_0,y_0)}{h}$$

gradient vector

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

$w = f(x,y,z)$ $\quad \vec{u} = \langle a,b,c\rangle$

$$D_{\vec{u}} f(x_0,y_0,z_0) = \nabla f(x_0,y_0,z_0)\cdot\vec{u}$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

examples (1) $f(x,y) = x^2+y^2$, $\quad$ (3) #9 $f(x,y,z) = x^2yz - xyz^3$, $P(2,-1,1)$, $\vec{u} = \langle 0, \frac{4}{5}, -\frac{3}{5}\rangle$

(2) $f(x,y) = xe^y + \cos(xy)$, $\quad$ (4) #16 $f(x,y,z) = \sqrt{xyz}$, $(3,2,6)$, $\vec{v} = \langle -1,-2,2\rangle$

Lesson 7

find the max. rate of change of f

(5) #22 $f(s,t) = t e^{st}$, $(0,2)$, (6) #24, $f(x,y,z) = \dfrac{x+y}{z}$, $(1,1,-1)$

- gradients and tangents to level sets (curves or surfaces)

level surface S: $f(x,y,z) = k$

any smooth curve C on S : $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$\Rightarrow f(\vec{r}(t)) = k \Rightarrow \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$$

$$\Rightarrow \boxed{\nabla f \perp S = \{(x,y,z) \mid f(x,y,z) = k\}}$$

- equation of tangent plane at $(x_0, y_0, z_0) \in S$

(1) level surface $f(x,y,z) = k$

$$0 = \nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0)$$

(2) graph $z = f(x,y) \iff 0 = f(x,y) - z \equiv g(x,y,z)$

$$0 = \nabla g(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0)$$

$$= \left\langle \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1 \right\rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$$

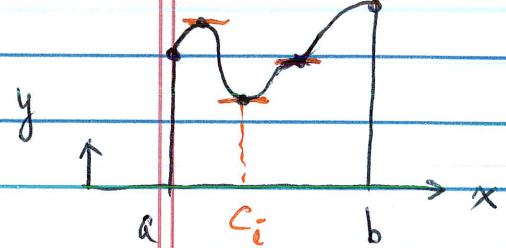
~~examples~~

- equation of normal line at $(x_0, y_0, z_0) \in S$

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \nabla f(x_0, y_0, z_0)$$

examples #43 $xyz^2 = 6$, $(3,2,1)$

§14.7 Maximum and Minimum Values (Lesson 8: beg.–Ex. 4; Lesson 9: Ex. 5–Ex. 7)



$$\max_{x\in[a,b]}/\min f(x) = \max/\min\{f(a), f(b), f(c_i)\}$$

$f(c_i)=0$ — c_i – critical pt

$\{a,b\}$ – boundary pts, $\{c_i\}$ – critical pts (interior)

Def. (1) $f(a,b)$ is a local maximum value of $f \iff f(a,b) \geq f(x,y)$ $\forall (x,y)\in D_r(a,b)$

(2) - - - - - - minimum - - - - $\iff f(a,b) \leq f(x,y)$ $\forall (x,y)\in D_r(a,b)$

Thrm (1st Der. Test) Assume that

(1) f has a local max/min at an interior pt (a,b)

(2) f_x and f_y exist

$\Longrightarrow$ "$f_x(a,b)=0$ and $f_y(a,b)=0 \iff \nabla f(a,b)=\langle 0,0\rangle$"

Proof Assume that f has a local max at (a,b)

$\Rightarrow \forall (h_1,h_2)\in R^2$, $g(t)=f((a,b)+t(h_1,h_2))$ has a local max at $t=0$

$\Rightarrow 0=g'(0)=\nabla f(a,b)\cdot\langle h_1,h_2\rangle \Longrightarrow \nabla f(a,b)=\langle 0,0\rangle$. #

Def. (1) critical pts (a,b): (1) (a,b) is an interior pt ~~and (2) $\nabla f(a,b)$~~

(2) $\nabla f(a,b)=\langle 0,0\rangle$ or DNE.

(2) saddle pts (a,b): (a,b) is a critical pt but local max/min.

example 1 $f(x,y)=x^2+y^2-2x-6y+14$

$\nabla f = \langle 2x-2,\ 2y-6\rangle = \langle 0,0\rangle \Longrightarrow (x,y)=(1,3)$

$f(x,y)=4+(x-1)^2+(y-3)^2 \geq 4 = f(1,3)$ #

2. Find the extreme values of $f(x,y)=y^2-x^2$

$\nabla f = \langle -2x,\ 2y\rangle = \langle 0,0\rangle \Longrightarrow (x,y)=(0,0)$

$f(x,0)=-x^2\leq 0$ and $f(0,y)=y^2\geq 0 \Longrightarrow f(0,0)=0$ is not extreme value of f.

Thrm (2^{nd} Der. Test) Assume that

(1) $f_{xx}, f_{yy}, f_{xy} \in C^0(D_r(a,b))$

(2) $\nabla f(a,b)=\langle 0,0\rangle$

$\Longrightarrow$ (1) $D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy}\end{vmatrix}_{(a,b)} > 0$ $\begin{cases} f_{xx}(a,b)<0 \Longrightarrow f(a,b) \text{ is a local max} \\ f_{xx}(a,b)>0 \Longrightarrow \quad " \quad " \quad " \quad " \quad \text{min}\end{cases}$

(2) $D<0 \Longrightarrow (a,b)$ is a saddle pt

(3) $D=0$, the test is inconclusive.

examples 3. Find the l. max. and min values and saddle pts of

$$f(x,y)=x^4+y^4-4xy+1$$

4. Find and classify the critical pts of

$$f(x,y)=10x^2y-5x^2-4y^2-x^4-2y^4$$

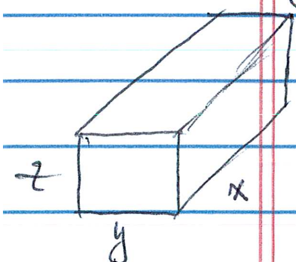
Lesson 9

examples

5. Find the shortest distance from $(1,0,-2)$ to the plane $x+2y+z=4$

$$d(x,y) = \sqrt{(x-1)^2+y^2+(z+2)^2} = \sqrt{(x-1)^2+y^2+(6-x-2y)^2}$$

6. A rectangular box without a lid is to be made from $12\,m^2$ of cardboard. Find the max. volume of such a box.



$$V = xyz \quad \text{and} \quad 12 = xy + 2xz + 2yz$$

$$\Downarrow$$

$$z = (12-xy)\Big/(2x+2y)$$

Absolute Max. and Min. Values

Thrm f is a cont. on a closed, bounded set $D \subset R^2$

$\Rightarrow$ f attains an abs. max value $f(x_1, y_1)$

and " " min " $f(x_2, y_2)$ in D.

Algorithm (1) calculate values of f at critical pts of D

(2) " extreme values of on ∂D — boundary of D

(3) abs. max. value = the largest of the values in (1) and (2)

" min. " = " smallest - - - - - - - - - .

example 7 $f(x,y) = x^2 - 2xy + 2y$ on $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

§14.8 Lagrange Multipliers

Constrained Maxima/Minima

Ex. 1 Find (x,y,z) on the plane $2x+\overset{\overset{g(x,y,z)}{\shortparallel}}{y}-z=5$ that is closest to $(0,0,0)$.

Solution

$$\min_{(x,y,z)} \sqrt{x^2+y^2+z^2} \Leftrightarrow \min_{(x,y,z)} \left(x^2+y^2+z^2\right)=f(x,y,z)$$

$g(x,y,z)=c$ $\qquad$ $g(x,y,z)=c$

$$\Leftrightarrow \min f\Big|_{g=c}$$

Ex. 0 $\min\left(x^2+y^2\right)$

$2x+y=5$

$$z=2x+y-5$$

$$h(x,y)=f\Big|_{g=c}=x^2+y^2+\left(2x+y-5\right)^2$$

Ex. 2 Find (x,y,z) on $g(x,y,z)=x^2-z^2-1=0$ that are closest to $(0,0,0)$

Solution 1

$$\min_{g=0} f(x,y,z) \Longleftrightarrow \min f\Big|_{g=0}$$

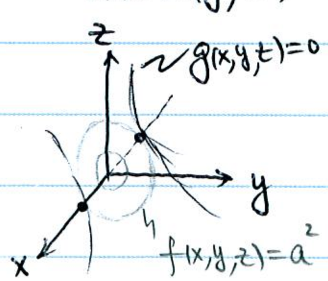
- $h(x,y)=f(x,y,z)\Big|_{z^2=x^2-1}=x^2+y^2+\left(x^2-1\right)$

  $\min h(x,y) \Longrightarrow$ critical pt $(x,y)=(0,0) \Rightarrow z^2=0-1=-1$ wrong

- $k(y,z)=f(x,y,z)\Big|_{x^2=z^2+1}=\left(z^2+1\right)+y^2+z^2 \geq 1$ — min at $(\pm1,0,0)$

  $\min k(y,z) \Longrightarrow$ critical pt $(y,z)=(0,0) \Rightarrow x^2=1 \Rightarrow x=\pm1$

Solution 2



$$\begin{cases} \nabla f=\lambda\nabla g \\ x^2-z^2-1=0 \end{cases} \Longrightarrow \begin{cases} 2x=2\lambda x \\ 2y=0 \\ 2z=2\lambda z \\ x^2-z^2-1=0 \end{cases} \Longrightarrow \overset{\lambda=1}{(\pm1,0,0)}$$

The Method of Lagrange Multipliers

Assume that (1) $f(x,y,z)$ and $g(x,y,z)$ are diff.

(2) $\nabla g \neq 0$ when $g = 0$

critical pts for $\max/\min\limits_{g=0} f$ satisfies

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$

Ex. 3 $\max/\min f(x,y) = xy$

$1 = g(x,y) = \frac{x^2}{8} + \frac{y^2}{2}$

· Geometry of the solution

Ex. 4 $\max/\min f(x,y) = 3x + 4y$

$g(x,y) \equiv x^2 + y^2 = 1$

Lagrange Multipliers with 2 Constraints

$\max/\min f(x,y,z)$

$g_1(x,y,z) = 0$

$g_2(x,y,z) = 0$

$$\begin{cases} \nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \\ g_1 = 0 \\ g_2 = 0 \end{cases}$$

Ex. 5