Chapter 14 Partial Derivatives (10 lectures)

§14.1 Functions of Several Variables

- functions of two variables

$$z = f(x, y)$$

dep. var. (z)　　indep. variables (x, y)

domain $D =$

range $R =$

$\forall (x,y) \in D$, $f(x,y)$ has a single value

Ex. 1 (a) $f(x,y) = \dfrac{\sqrt{x+y-1}}{x-1}$; (b) $f(x,y) = x \ln(y^2 - x)$; (c) $f(x,y) = \dfrac{\sqrt{y-x^2}}{1-x^2}$

evaluate $f(3,2)$, find and sketch the domain

• <u>graphes of $z = f(x,y)$</u>

$$S = \left\{ (x, y, f(x,y)) \;\middle|\; (x,y) \in D \right\}$$

<u>examples</u> sketch the graphes of

(a) $f(x,y) = 6 - 3x - 2y$; (b) $g(x,y) = \sqrt{9 - x^2 - y^2}$; (c) $h(x,y) = 4x^2 + y^2$

- <u>level curves</u>

$$C = \{ (x,y) \mid f(x,y) = k \} \qquad k \text{ — constant}$$

<u>examples</u> sketch level curves of

(a) $f(x,y) = 6 - 3x - 2y$ for $k = -6, 0, 6, 12$; (b) $g(x,y) = \sqrt{9 - x^2 - y^2}$ for $k = 0, 1, 2, 3$

(c) $h(x,y) = 4x^2 + y^2 + 1$ for $k = 2, 3$.

• <u>functions of three variables</u>

$$w = f(x, y, z)$$

domain $D =$

graph $\Omega =$

level surface $S =$

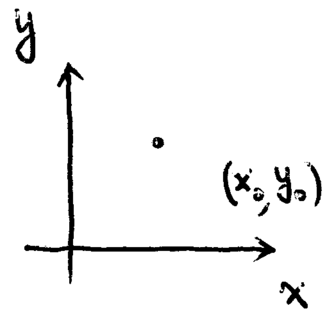
<u>ex. 14</u> $f(x,y,z) = \ln(z-y) + xy\sin z$, $D = ?$

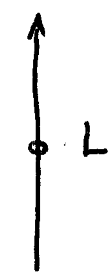
<u>ex. 15</u> $f(x,y,z) = x^2+y^2+z^2$, find the level surfaces.

§14.2 Limits and Continuity

- **limits**

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \iff$$





$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \text{ DNE} \iff$$

Ex. 1 Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ DNE.

examples Do the following limits exist?

(1) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$, (2) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^4+y^4}$, (3) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

- <u>continuity</u>

$f(x,y)$ is continuous at $(a,b) \Longleftrightarrow$

- polynomials and rational functions are continuous on their domain.

<u>ex. 5</u> $\lim\limits_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) =$

<u>ex. 6</u> Where is the function $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ continuous?

<u>ex. 7</u> Is $g(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ continuous at $(0,0)$?

Ex. 8 Is $f(x,y)=\begin{cases} \dfrac{3x^2y}{x^2+y^2}, & \text{if } (x,y)\neq(0,0) \\ 0, & \text{if } (x,y)=(0,0) \end{cases}$ continuous at $(0,0)$?

Ex. 9 Where is $h(x,y)=\arctan\left(\dfrac{y}{x}\right)$ continuous?

- **Functions of three or more variables**

limit $\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)=L$

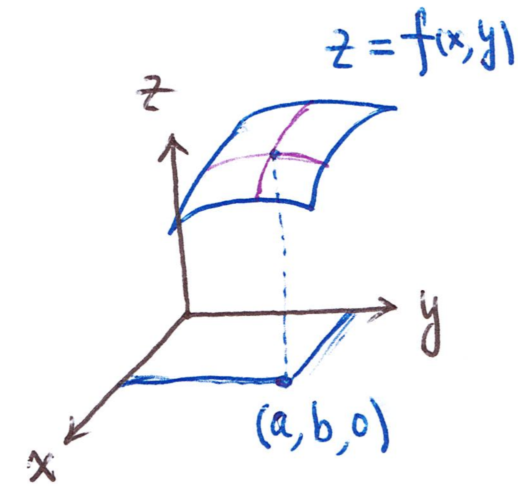
continuity $f(x,y,z)$ is cont. at $(x_0,y_0,z_0) \Longleftrightarrow$

§14.3 Partial Derivatives

$z = f(x, y)$

$$\frac{\partial f}{\partial x}(a, b) =$$

$$\frac{\partial f}{\partial y}(a, b) =$$



Examples

(1) $f(x, y) = x^2 + 3xy + y - 1$, $\frac{\partial f}{\partial x}(4, -5) = ?$ $\frac{\partial f}{\partial y}(4, -5) = ?$

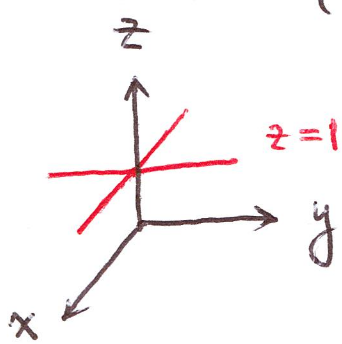
(2) $f(x, y) = y \sin(xy)$, $\frac{\partial f}{\partial x} = ?$ $\frac{\partial f}{\partial y} = ?$

(3) $f(x,y) = \dfrac{2y}{y+\cos x}$, $f_x = ?$, $f_y = ?$

(4) $yz - \ln z = x + y$, $\dfrac{\partial z}{\partial x} = ?$, $\dfrac{\partial z}{\partial y} = ?$

- <u>partial derivatives and continuity</u>

$$f(x,y) = \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$



- $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) =$
- $\dfrac{\partial f}{\partial x}(0,0) =$
- $\dfrac{\partial f}{\partial y}(0,0) =$

- higher-order derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \cdots$$

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right), \quad \cdots$$

examples (1) $f(x,y,z) = 1 - 2xy^2z + x^2y$, $\frac{\partial^4 f}{\partial y \partial x \partial y \partial z} = ?$

(2) $w = \sqrt{u^2+v^2}$, $\frac{\partial^2 w}{\partial u^2} = ?$ $\frac{\partial^2 w}{\partial u \partial v} = ?$

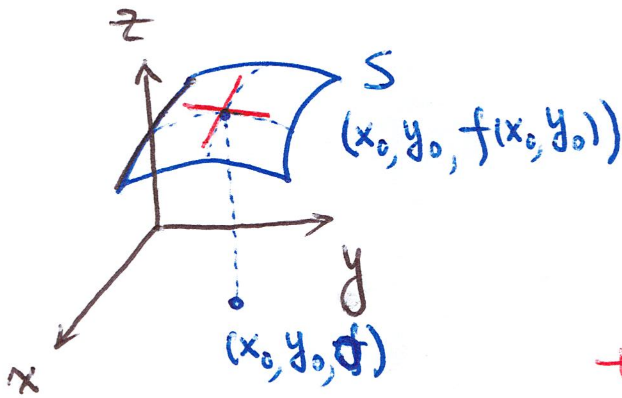
Clairaut's Thrm $f(x,y)$ is defined on a disk D and $(a,b) \in D$

f_{xy} and f_{yx} are cont. on D $\Longrightarrow$ $f_{xy}(a,b) = f_{yx}(a,b)$

§14.4 Tangent Planes and Linear Approximations

∘ <u>tangent planes</u>

Surface S : $z = f(x, y)$ has cont. partial derivatives



the equation of tangent plane to S at $\big(x_0, y_0, \overset{z_0}{\overset{\|}{f(x_0,y_0)}}\big)$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

the eq of any plane passing through the point (x_0, y_0, z_0)

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$\Rightarrow \quad z - z_0 = -\frac{A}{C}(x - x_0) - \frac{B}{C}(y - y_0)$$

<u>when $y = y_0$</u> $\quad z - z_0 = -\dfrac{A}{C}(x - x_0) \xrightarrow[\text{with slope } f_x(x_0,y_0)]{\text{line eq}} -\dfrac{A}{C} = f_x(x_0, y_0)$

examples Find an equation of the tangent plane to the given surface at the specified point

(1) $z = 3y^2 - 2x^2 + x$ at $(2,-1,-3)$

(2) $z = x e^{xy}$ at $(2,0,2)$

- Linear Approximations for (x, y) near (x_0, y_0)

$$f(x,y) \approx L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Linearization of f
or linear approximation of f

example $f(x,y) = 2x^2 + y^2$ near $(1, 1, 3)$

$$f(x,y) \approx L(x,y) =$$

$$f(1.1, 0.95) \approx$$

- <u>differentiability</u>

$f(x)$ is differentiable at $x_0 \Longleftrightarrow$

$f(x,y)$ is differentiable at $(x_0, y_0) \Longleftrightarrow$

<u>Thrm</u> If f_x and f_y exist near (x_0, y_0) and are continuous at (x_0, y_0)

$\Longrightarrow f$ is differentiable at (a, b)

<u>Ex. 2</u> Show that $f(x,y) = xe^{xy}$ is diff. at $(1,0)$, ~~and~~ find its linearization, and use it to approximate $f(1.1, -0.1)$.

• <u>differentials</u> (linear approx. to the change in function)

<u>$y=f(x)$</u> $\Delta y = f(x+\Delta x) - f(x) \approx$

<u>$z=f(x,y)$</u> $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) \approx$

<u>differential</u> $dz =$

<u>Ex. 4</u> $z = f(x,y) = x^2 + 3xy - y^2$

(a) find the differential dz, (b) compare the values of Δz and dz for changes from $(2,3)$ to $(2.05, 2.96)$

Ex. 5 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

$$V = \frac{\pi}{3} r^2 h$$

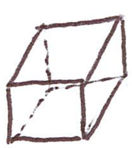
- Functions of three or more variables $w = f(x, y, z)$

$$f(x,y,z) \approx f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

increment $\Delta w = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$

differential $dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$

Ex. 6 $V = xyz$, $(75, 60, 40)$, $|\Delta x| \leq 0.2$, $|\Delta y| \leq 0.2$, $|\Delta z| \leq 0.2$



$\Delta V \approx dV =$

§14.5 Chain Rule

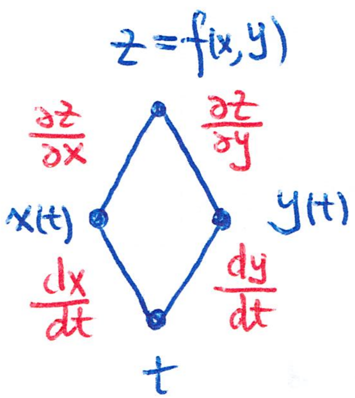
- $\begin{cases} y = f(x) \\ x = g(t) \end{cases} \Longrightarrow y = f(g(t))$

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = f'(g(x))\, g'(x)$$

- $\begin{cases} z = f(x,y) \\ x = g(t) \\ y = h(t) \end{cases} \Longrightarrow z = f(g(t), h(t))$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$



dep. var.

intermediate var.

indep. var.

$$\frac{dz}{dt} = \lim_{h\to 0} \frac{f(g(t+h), h(t+h)) - f(g(t), h(t))}{h}$$

$$= \lim_{h\to 0} \frac{f_x(g(t), h(t))\,(g(t+h) - g(t)) + f_y(g(t), h(t))\,(h(t+h) - h(t))}{h}$$

differentiability

Examples

(1) $z = \cos(x+4y)$, $\begin{cases} x = 5t^4 \\ y = \dfrac{1}{t} \end{cases}$, $\dfrac{dz}{dt} = ?$

(2) The pressure P (in kilopascals), volume V (in liters), and temperature (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

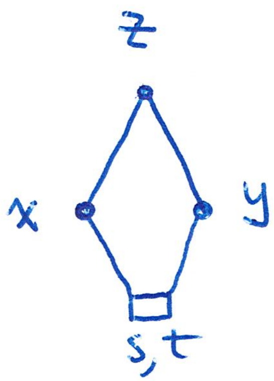
Given

Unknown

- $\begin{cases} z = f(x,y) \\ x = g(s,t) \\ y = h(s,t) \end{cases} \Longrightarrow z = f\big(g(s,t),\, h(s,t)\big)$

$$\frac{\partial z}{\partial s} =$$

$$\frac{\partial z}{\partial t} =$$



dep. var.

intermediate var.

indep. var.

<u>examples</u>

(1) $z = x^2 + y^2, \quad \begin{cases} x = s - t \\ y = s + t \end{cases}$

$$\frac{\partial z}{\partial s} =$$

$$\frac{\partial z}{\partial t} =$$

(2) $z = e^r \cos\theta, \quad \begin{cases} r = st \\ \theta = \sqrt{s^2 + t^2} \end{cases}$

$$\frac{\partial z}{\partial s} =$$

$$\frac{\partial z}{\partial t} =$$

- $\begin{cases} u = f(x_1, \cdots, x_n) \\ x_1 = x_1(t_1, \cdots, t_m) \\ \vdots \\ x_n = x_n(t_1, \cdots, t_m) \end{cases} \Longrightarrow u = f\big(x_1(t_1, \cdots, t_m), \cdots, x_n(t_1, \cdots, t_m)\big)$

$$\frac{\partial u}{\partial t_i} =$$

Examples (1) $w = f(x, y, z, t)$

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \\ t = t(u, v) \end{cases}$$

$$\frac{\partial w}{\partial u} =$$

$$\frac{\partial w}{\partial t} =$$

(2) $W = xy + yz + zx$ $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r\theta \end{cases}$

$$\frac{\partial W}{\partial r}\left(2, \frac{\pi}{2}\right) =$$

$$\frac{\partial W}{\partial \theta}\left(2, \frac{\pi}{2}\right) =$$

(3) $g(s,t) = f(s^2 - t^2,\ t^2 - s^2)$. Show that $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$

(4) $z = f(x,y)$, $\begin{cases} x = r^2 + s^2 \\ y = 2rs \end{cases}$

$\frac{\partial z}{\partial r} =$

$\frac{\partial^2 z}{\partial r^2} =$

$F(x, y, z) = 0$

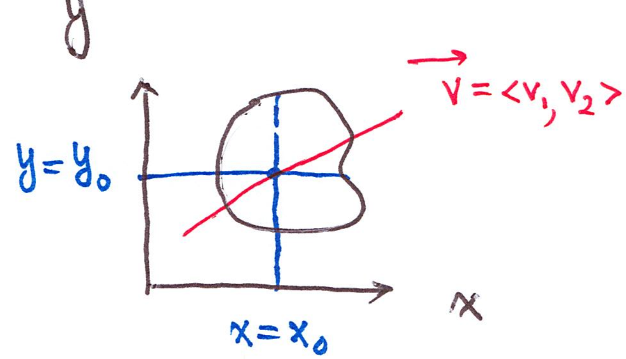
$$\frac{\partial z}{\partial x} =$$

$$\frac{\partial z}{\partial y} =$$

<u>example (#31)</u> $x^2 + 2y^2 + 3z^2 = 1$, $\frac{\partial z}{\partial x} = ?$, $\frac{\partial z}{\partial y} = ?$

§14.6 Directional Derivative and the Gradient Vector

$z = f(x, y)\,, \quad (x, y) \in D$



partial derivative

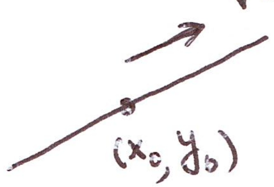
$\frac{\partial f}{\partial x}(x_0, y_0) =$

$\frac{\partial f}{\partial y}(x_0, y_0) =$

directional derivative along $\vec{v}$

the eq of the line

$\vec{v} = \langle v_1, v_2 \rangle$



$\vec{r}(t) =$

the restriction of f on the line

$f(\vec{r}(t)) =$

$\frac{d}{dt} f(\vec{r}(t)) \Big|_{t=0}$

$=$

- <u>directional derivative along the direction $\vec{v} = \langle v_1, v_2 \rangle$</u>

$$D_{\vec{v}} f(x_0, y_0) = \frac{d}{dt} f(\vec{r}(t)) \Big|_{t=0}$$

$$= \nabla f(x_0, y_0) \cdot \vec{v}$$

$\vec{v} = \vec{i} = \langle 1, 0 \rangle$ or $\vec{v} = \vec{j} = \langle 0, 1 \rangle$

- <u>gradient vector</u>

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

- <u>functions of three variables</u>

$w = f(x, y, z)$, $\nabla f = \langle \qquad \rangle$, $D_{\vec{v}} f(x_0, y_0, z_0) =$

<u>examples</u> Find the gradient of, directional derivative along $\vec{u}$

(1) $f(x,y) = xe^{y} + \cos(xy)$, $P(2,0)$, $\vec{u} = <3,-4>$

(2) (#9) $f(x,y,z) = x^{2}yz - xyz^{3}$, $P(2,-1,1)$, $\vec{u} = <0, \frac{4}{5}, -\frac{3}{5}>$

<u>Question</u> In which direction at (x_0, y_0) does the value of $f(x,y)$ increase most rapidly?

$\Longleftrightarrow$ Find $\vec{u} = \langle a, b \rangle$ such that

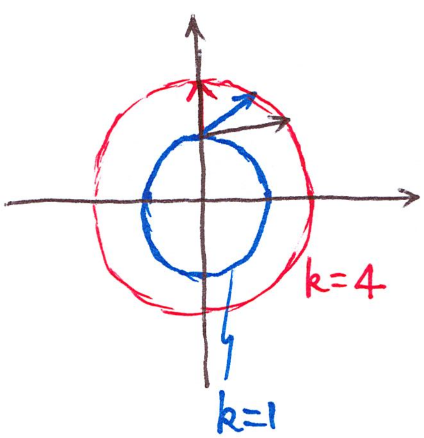
$$\frac{d}{dt} f\left(\langle x_0, y_0 \rangle + t\vec{u}\right)\Big|_{t=0} = \max_{\substack{\vec{v} \in \mathbb{R}^2 \\ |\vec{v}|=1}} \frac{d}{dt} f\left(\langle x_0, y_0 \rangle + t\vec{v}\right)\Big|_{t=0}$$

$$= \max_{|\vec{v}|=1} \left\{ \nabla f(x_0, y_0) \cdot \vec{v} \right\}$$

$$=$$

<u>example</u> $z = f(x,y) = x^2 + y^2$

<u>level curve</u> $x^2 + y^2 = k$



Theorem Assume that f is differentiable

$$\Longrightarrow \max_{|\vec{v}|=1} D_{\vec{v}} f(x,y) = |\nabla f|$$

it occurs when $\vec{v}$ is in the same direction as ∇f.

examples find the maximum rate of change of f at the given point and the direction in which it occurs.

(3) (#22) $f(s,t) = te^{st}$, $(0,2)$

(4) (#24) $f(x,y,z) = \dfrac{x+y}{z}$, $(1,1,-1)$

- gradients and tangents to level sets (curves or surfaces)

level surface $\quad S = \{(x,y,z) \mid f(x,y,z) = k\}$

$\forall$ smooth curve C on S : $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$\boxed{\nabla f \perp S}$$

$\Rightarrow f(\vec{r}(t)) = k$

$\Rightarrow 0 = \frac{d}{dt} f(\vec{r}(t)) =$

- equations of tangent planes at $(x_0, y_0, z_0) \in S$

(1) level surface $\quad f(x,y,z) = k$

(2) graph $\quad z = f(x,y)$

- equations of normal lines at $(x_0, y_0, z_0) \in S$

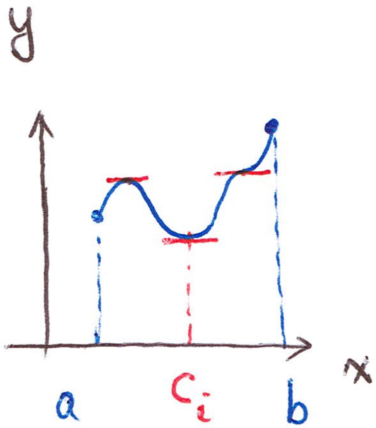
$$\vec{r}(t) =$$

examples find the equations of the tangent plane and normal line

(1) $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 3$; $P(-2, 1, -3)$

(2) $xyz^2 = 6$, $P(3, 2, 1)$

§14.7 Maximum and Minimum Values



$$\max_{x\in[a,b]}/\min f(x) = \max/\min \{f(a), f(b), f(c_i)\}$$

$f(c_i) = 0$, c_i —

$\{a, b\}$ — boundary points

$\{c_i\}$ — critical points (interior)

Def. (extreme values)

(1) $f(a,b)$ is a local maximum value of f $\Longleftrightarrow$

(2) $f(a,b)$ is a local minimum value of f $\Longleftrightarrow$

Thrm (1st derivative test)

f has a local maximum/minmum at an interior point (a,b) and $f_x(a,b)$ and $f_y(a,b)$ exist $\Longrightarrow$ $\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle = \langle 0, 0 \rangle$

Proof Assume that $f(a,b)$ is a local maximum value of f

$\Longrightarrow$

Def.

- (a,b) is a critical point of f $\Longleftrightarrow$ $\nabla f(a,b) = \langle 0, 0 \rangle$
- (a,b) is a saddle point of f $\Longleftrightarrow$ $\nabla f(a,b) = \langle 0, 0 \rangle$ but $f(a,b)$ is not a local max./min.

<u>examples</u> Find extreme values

(1) $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

(2) $f(x,y) = y^2 - x^2$

Thrm (2nd derivative test)

Assume that f_{xx}, f_{yy}, and f_{xy} are continuous on a disk with center (a,b)

and that $\nabla f(a,b) = \langle 0, 0 \rangle$.

Let $D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$.

$\Longrightarrow$ (a) $D > 0$ and $f_{xx}(a,b) > 0 \Longrightarrow f(a,b)$ is a local ~~maximum~~ minimum.

(b) $D > 0$ and $f_{xx}(a,b) < 0 \Longrightarrow f(a,b)$ is a local maximum.

(c) $D < 0 \Longrightarrow f(a,b)$ is not a local max./min.

Ex. 3 Find the local max. and min. values and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$

Ex. 4 Find and classify the critical points of $f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$

Also find the heighest point on the graph of f.

Ex. 5 Find the shortest distance from the point $(1,0,-2)$ to the plane $x + 2y + z = 4$.

Ex. 6 A rectangular box without a lid is to be made from $12\ m^2$ of cardboard. Find the maximum volume of such a box.

- Absolute Maximum and Minimum Values

Thrm

f is continuous on a closed, bounded set $D \subset \mathbb{R}^2$ $\Longrightarrow$ f attains an abs. max. value and an abs. min. value in D.

Ex. 7 Find the absolute maximum and minimum values of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y) \mid 0 \le x \le 3, 0 \le y \le 2\}$.

§14.8 Lagrange Multipliers

- constrained maximization and minimization

$$\max / \min \ f(x,y,z)$$
$$g(x,y,z) = k$$

Ex. 5 (§14.7) Find the shortest distance from the pt $(1,0,-2)$ to the plane $x+2y+z=4$

Ex. 6 (§14.7) A rectangular box without a lid is to be made from $12\ m^2$ of cardboard.
Find the maximum volume of such a box.

$$\max/\min \; f(x,y,z)$$
$$g(x,y,z)=k$$

- method of Lagrange multipliers

Assume that the extreme values of f exist and that $\nabla g \neq \vec{0}$ on the surface $g(x,y,z)=k$.

(a) Find all values of x, y, z, and λ such that

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = k \end{cases}$$

(b) Evaluate f at all points (x,y,z) resulting from step (a).

- The largest of these values is the maximum value of f.
- The smallest of these values is the minimum value of f.

<u>Ex. 1</u> A rectangular box without a lid is to be made from 12 m^2 of cardboard.

Find the maximum volume of such a box.

Ex. 2 Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2+y^2=1$.

Ex. 3 Find the extreme values of $f(x,y) = x^2+2y^2$ on the disk $x^2+y^2 \leq 1$.

Ex. 4 Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,1,-1)$.

§14.8 Constrained Optimization and Lagrange Multipliers

$$f : U \subset \mathbb{R}^n \to \mathbb{R}^n \quad \text{and} \quad g : U \subset \mathbb{R}^n \to \mathbb{R}^n$$

$$S = \{ \vec{x} \in U \mid g(\vec{x}) = k \} \text{ — level set}$$

Thrm (method of Lagrange multiplier)

$f|_S$ has a local extreme values at $\vec{x}_0 \in S$ and $\nabla g(\vec{x}_0) \neq \vec{0}$

critical pt of $f|_S$

$$\Longrightarrow \exists\ \lambda \in \mathbb{R} \text{ s.t. } \nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0)$$

Lagrange multiplier

Sketch of Proof

- $\nabla g(\vec{x}_0) \perp S$ at $\vec{x}_0 \Longleftrightarrow \forall\ \vec{r}(t)$ on S with $\vec{r}(0) = \vec{x}_0$, $\nabla g(\vec{x}_0) \cdot \vec{r}\,'(0) = 0$
- $f|_S$ has a local extreme values at $\vec{x}_0 \Rightarrow f(\vec{r}(t))$ has a local extreme value at $t=0$

$$\Longrightarrow 0 = \frac{d}{dt} f(\vec{r}(t))\Big|_{t=0} = \nabla f(\vec{x}_0) \cdot \vec{r}\,'(0) \Longrightarrow \nabla f(\vec{x}_0) \perp S \text{ at } \vec{x}_0$$

Ex.1 Find the extreme values of $f(x,y) = x^2+y^2$ on the line $y-x+1=0$

Ex.2 Find the extreme values of $f(x,y) = x^2 - y^2$ on the circle $x^2 + y^2 = 1$

Ex. 3 Maximize $f(x,y,z) = x+z$ subject to the constraint $x^2+y^2+z^2=1$.