Chapter 15 Multiple Integrals (7 lessons)

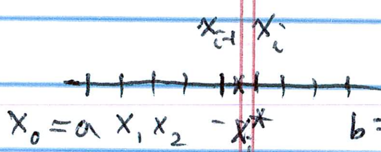
(Lesson 1 §15.1 – 15.2)

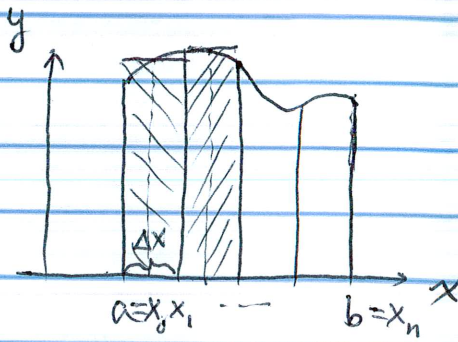
§15.1 Double Integrals over Rectangles

- Review of the Definite Integral

$I = [a, b]$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$





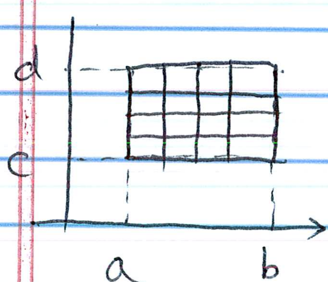
$\Delta x = \frac{b-a}{n}$

- Volumes and Double Integrals

$R = [a, b] \times [c, d]$

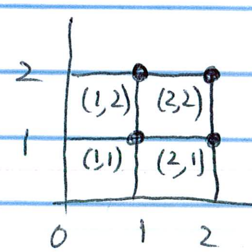
$a = x_0 < x_1 < \cdots < x_m = b$ $\qquad \Delta x = \frac{b-a}{m}$

$c = y_0 < y_1 < \cdots < y_n = d$ $\qquad \Delta y = \frac{d-c}{n}$

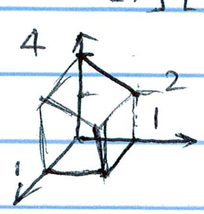


$$V = \iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\underbrace{x_{ij}^*, y_{ij}^*}_{\text{sample pt}})\,\Delta A$$

example 1 Estimate $\displaystyle\iint_{[0,2]\times[0,2]} (16 - x^2 - 2y^2)\,dA$



example 2 (#13) $\displaystyle\iint_{[0,1]\times[0,1]} (4 - 2y)\,dA$ as a volume of a solid



$$= 1 \times \frac{2+4}{2} \cdot 1$$

§15.2 Iterated Integrals

Fubini's Thrm $\displaystyle\iint_{[a,b]\times[c,d]} f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$

examples 2. $\displaystyle\iint_{[0,2]\times[1,2]} (x-3y^2)\,dA$; 3. $\displaystyle\iint_{[1,2]\times[0,\pi]} y\sin(xy)\,dA$

4. Find the volume of the solid S that is bounded by $x^2+2y^2+z=16$, $x=2$, $y=2$, and the 3 coordinate planes.

$$\iint_{R=[a,b]\times[c,d]} g(x)h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

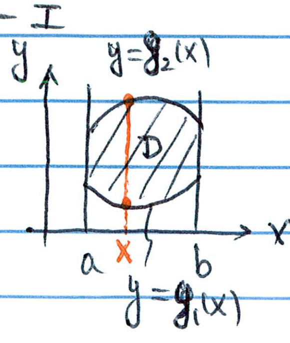
5. $\displaystyle\iint_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} \sin x\cos y\,dA$

(Lesson 2) §15.3 Double Integrals over General Regions

elementary regions

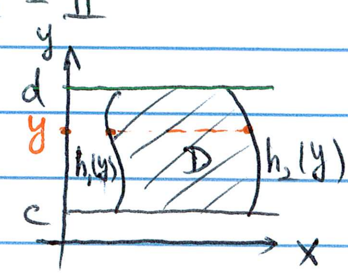
type - I

$$\begin{cases} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{cases}$$

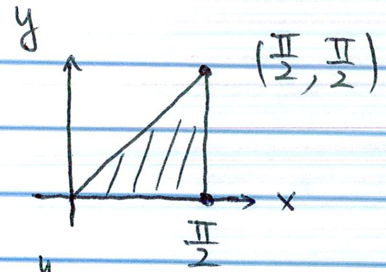


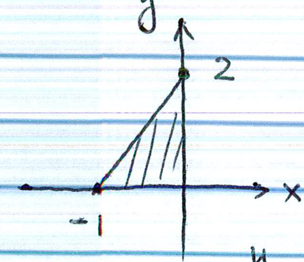
type - II

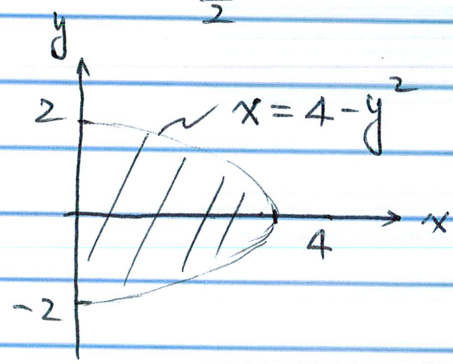
$$\begin{cases} h_1(y) \le x \le h_2(y) \\ c \le y \le d \end{cases}$$

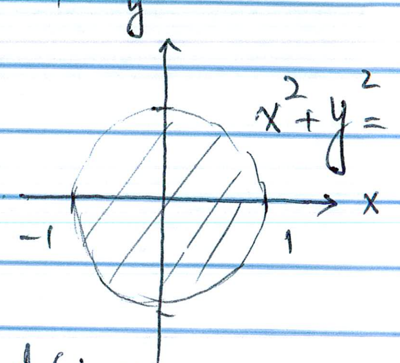


examples





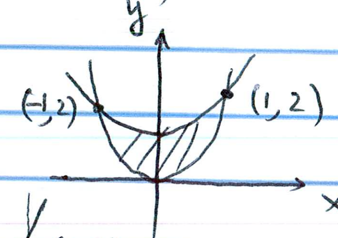




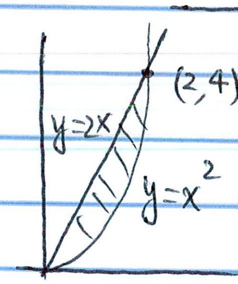
$$\iint_D f\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f\,dy \right) dx = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f\,dx \right) dy$$

examples

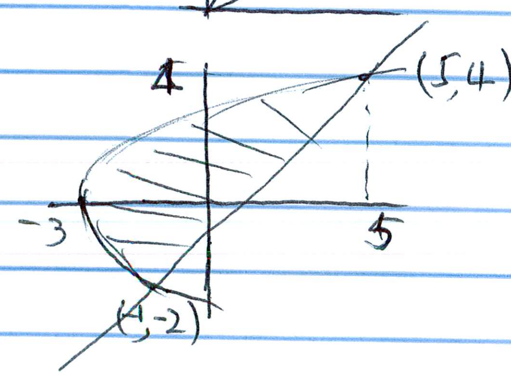
1. $\iint_D (x + 2y)\,dA$, $D: \begin{cases} y = 2x^2 \text{ and} \\ y = 1 + x^2 \end{cases}$



2. $\iint_D (x^2 + y^2)\,dA$, $D: \begin{cases} y = 2x \\ \text{and } y = x^2 \end{cases}$

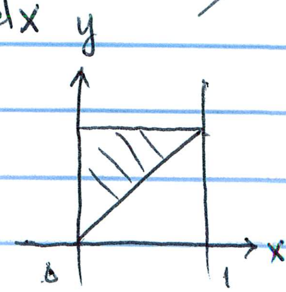


3. $\iint_D xy\,dA$, $D: \begin{cases} y = x - 1 \\ y^2 = 2x + 6 \end{cases}$



5. $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$

$D: \begin{cases} 0 \le x \le 1 \\ x \le y \le 1 \end{cases}$



$\begin{cases} 0 \le x \le y \\ 0 \le y \le 1 \end{cases}$

Properties of Double Integrals

- $\iint_D (f+g)\,dA = \iint_D f\,dA + \iint_D g\,dA$
- $\iint_D cf\,dA = c\iint_D f\,dA$
- $f(x,y) \geq g(x,y)\ \forall (x,y)\in D \Longrightarrow \iint_D f\,dA \geq \iint_D g\,dA$
- $\iint_{\substack{D=D_1\cup D_2 \\ D_1\cap D_2=\emptyset}} f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$
- $A(D) = \iint_D 1\,dA$
- $m \leq f(x,y) \leq M\ \forall (x,y)\in D \Longrightarrow m \leq \underbrace{\frac{1}{A(D)}\iint_D f\,dA}_{f_{ave}} \leq M$

(Lesson 3) §15.4 Double Integrals in Polar Coordinates

$$\iint_R f(x,y)\,dx\,dy = \iint_D f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

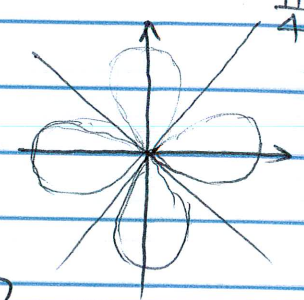
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \text{abs.}\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \text{abs.}\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

examples

1. $\iint_R (3x+4y^2)\,dA$, $R=\{(x,y)\mid y\ge 0 \text{ and } 1\le x^2+y^2\le 4\}$

2. Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.

3. use a double integral to find the area enclosed by one loop of the four-~~leaved~~ leaved rose $r=\cos 2\theta$



$D=\{(r,\theta)\mid -\frac{\pi}{4}\le\theta\le\frac{\pi}{4},\ 0\le r\le\cos 2\theta\}$

$A(D)=\iint_D dA$

4. Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$, above xy-plane, inside cylinder $x^2+y^2=2x$.

(Lesson 4 §15.5 —Ex.3 §15.6)

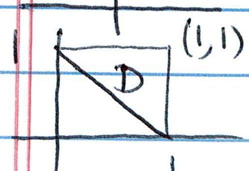
§15.5 Applications of Double Integrals

- Density and Mass

$$\overset{\text{mass}}{m}=\iint_D \overset{\text{density}}{\rho(x,y)}\,dA$$

total charge — charge density

example 1 $Q=\iint_D \sigma(x,y)\,dA=\iint_D xy\,dA$.



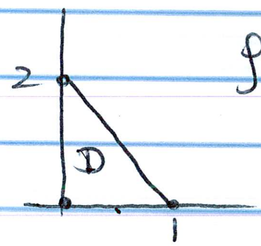
Moments and Centers of Mass

the moment about the x-axis $\quad M_x = \iint_D y\,\rho(x,y)\,dA$

" " " " y-axis $\quad M_y = \iint_D x\,\rho(x,y)\,dA$.

the center of mass $\quad (\bar{x}, \bar{y}) = \frac{1}{m}(M_y, M_x)$

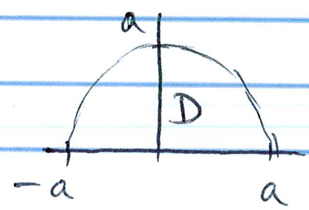
example 2



$\rho = 1 + 3x + y$

$m = ?$ $\quad (\bar{x}, \bar{y}) = ?$

example 3



$\rho(x,y) = K\sqrt{x^2+y^2}$, $\quad m = ?$

$(\bar{x}, \bar{y}) = ?$

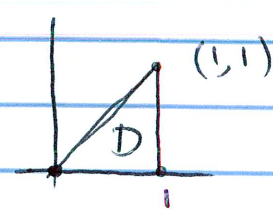
§15.6 Surface Area

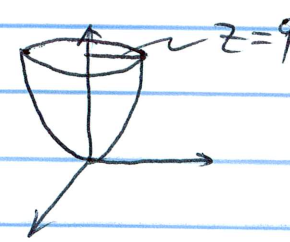
surface $\quad z = f(x,y)$, $(x,y) \in D$

$$A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

example 1 $\quad S: \; z = x^2 + 2y$

$A(S) = ?$





example 2 $\quad A(S) = ?$ $\quad S: \; z \geq x^2 + y^2$

$z \leq 9$

(Lesson 5) §15.7 Triple Integrals

Fubini's Thm on $B = [a,b] \times [c,d] \times [r,s]$

$$\iiint_B f \, dV = \int_r^s \int_c^d \int_a^b f \, dx \, dy \, dz$$

example 1 $\iiint_B xyz^2 \, dV$, $B = \{(x,y,z) \mid 0 \le x \le 1, \ -1 \le y \le 2, \ 0 \le z \le 3\}$

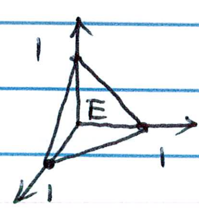
elementary regions

$\begin{cases} (x,y) \in D \\ u_1(x,y) \le z \le u_2(x,y) \end{cases}$, $\begin{cases} (y,z) \in D \\ u_1(y,z) \le x \le u_2(y,z) \end{cases}$, $\begin{cases} (x,z) \in D \\ u_1(x,z) \le y \le u_2(x,z) \end{cases}$

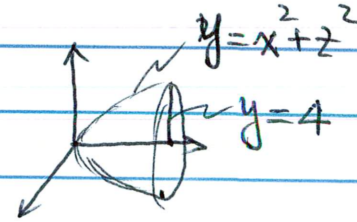
type-1, type-2, type-3

$$\iiint_E f(x,y,z) \, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f \, dz \right) dA = \cdots$$

examples 2. $\iiint_E z \, dV$



3. $\iiint_E \sqrt{x^2+z^2} \, dV$



4. $\int_0^1 \int_0^{x^2} \int_0^y f \, dz \, dy \, dx \xrightarrow{?} \iiint_E f \, dV \xrightarrow{?} \int dy \int dz \int f \, dx$

(Lesson 6) §15.8 Triple Integrals in Cylindrical Coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \\ z = z \end{cases}$$

examples (1) $(x, y, z) = (3, -3, -7)$

$$\left.\begin{aligned} r^2 &= 3^2 + (-3)^2 = 2\cdot 3^2 \Rightarrow r = 3\sqrt{2} \\ \tan\theta &= \frac{-3}{3} = -1 \Rightarrow \theta = \frac{7\pi}{4} + 2n\pi \end{aligned}\right\} \Rightarrow \left(3\sqrt{2}, \frac{7\pi}{4}, -7\right)$$

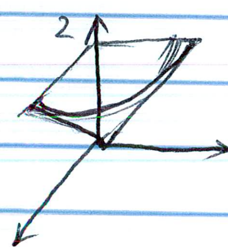
(2) $x^2 - x + y^2 + z^2 = 1 \Rightarrow r^2 - r\cos\theta + z^2 = 1$

$z^2 = 1 + r\cos\theta - r^2$

$z = x^2 - y^2 \Rightarrow z = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos(2\theta)$

(3) $0 \le \theta \le \frac{\pi}{2}$, $r \le z \le 2$

$z = r = \sqrt{x^2+y^2}$ — cone



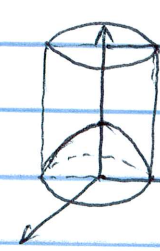
$$\iiint_E f(x,y,z)\,dV = \iiint_G f(r\cos\theta, r\sin\theta)\, z\; r\,dr\,d\theta\,dz$$

$$= \iint_D \left(\int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,dz \right) r\,dr\,d\theta$$

(4) E lies within the cylinder $x^2+y^2=1$ $\rightsquigarrow r=1$

below $z=4$, above $z=1-x^2-y^2$ $\rightsquigarrow z=1-r^2$

$$\iiint_E K\sqrt{x^2+y^2}\,dV$$



$1-r^2 \le z \le 4$

$0 \le r \le 1$

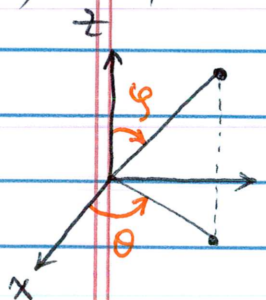
$0 \le \theta \le 2\pi$

(5) $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$$

$$r \le z \le 2$$

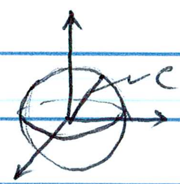
$$\left.\begin{array}{l}\sqrt{x^2+y^2} \le z \le 2\\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}\\ -2 \le x \le 2\end{array}\right\} \Rightarrow D = \left\{(x,y)\,\middle|\, 0 \le x^2+y^2 \le 4\right\} \quad \left\{\begin{array}{l}0 \le r \le 2\\ 0 \le \theta \le 2\pi\end{array}\right.$$

(Lesson 9) §15.9 Triple Integrals in Spherical Coordinates

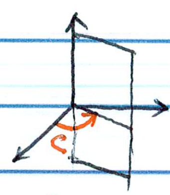


$P(\rho,\theta,\varphi)$

$\rho \ge 0, \quad 0 \le \varphi \le \pi, \quad 0 \le \theta \le 2\pi$



$\rho = c$
sphere



$\theta = c$
half-plane

$\varphi = c$
half-cone

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases} \qquad \rho^2 = x^2+y^2+z^2$$

examples $(\rho,\theta,\varphi) = \left(2, \frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow (x,y,z) = (0,2,0)$

$(x,y,z) = (1,0,\sqrt{3}) \Rightarrow \rho^2 = 1^2+0^2+(\sqrt{3})^2 = 4 \Rightarrow \rho = 2$

$\cos\varphi = \frac{z}{\rho} = \frac{\sqrt{3}}{2} \Rightarrow \varphi = \frac{\pi}{6}$

$\tan\theta = \frac{y}{x} = 0 \Rightarrow \theta = 0, \pi, 2\pi$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \text{abs.}\begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi} \end{vmatrix} = \text{abs.}\begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}$$

$$= \text{abs.}\left[\cos\varphi\begin{vmatrix} -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \end{vmatrix} - \rho\sin\varphi\cdot\rho\sin^2\varphi\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}\right]$$

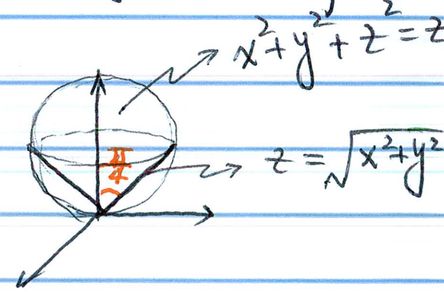
$$= \text{abs.}\left[-\cos\varphi\cdot\rho^2\sin\varphi\cos\varphi - \rho^2\sin\varphi\sin^2\varphi\right] = \rho^2\sin\varphi$$

$$\iiint_E f(x,y,z)\,dV = \iiint_G f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

<u>examples</u>

3. $\displaystyle\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dV$, $B = \left\{(x,y,z) \mid x^2+y^2+z^2 \le 1\right\}$

4. Volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.



#40 $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \left(x^2z + y^2z + z^3\right)dz\,dx\,dy$

#25 $\displaystyle\iiint_E x e^{x^2+y^2+z^2}\,dV$ E: the portion of the unit ball $x^2+y^2+z^2 \le 1$ that lies in the 1st octant.