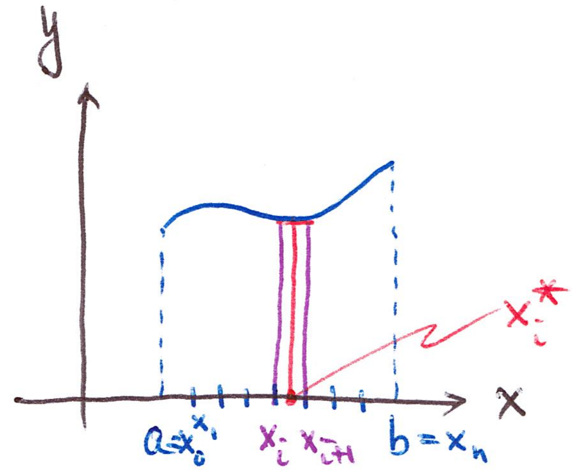
Chapter 15 Multiple Integrals (7 lectures)

§15.1 Double Integrals over Rectangles

- <u>Review of the Definite Integral</u>

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_i f(x_i^*)\,\Delta x$$

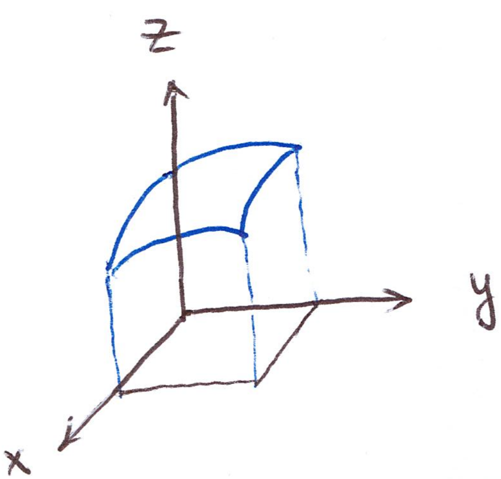


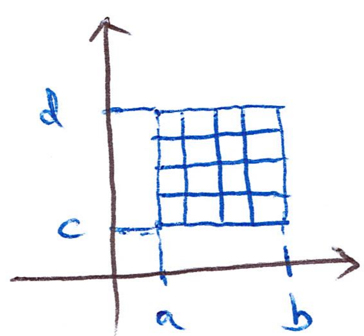
$$\Delta x = \frac{b-a}{n}$$

a partition of interval $[a,b]$: $x_i = x_0 + i\,\Delta x$

- <u>Volume and Double Integral</u>

a partition of rectangle $[a,b]\times[c,d]=R$





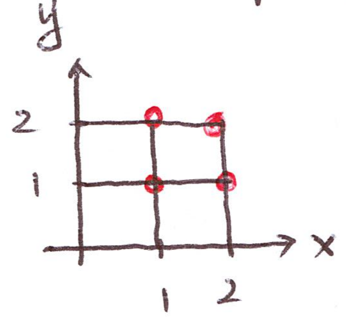
$a = x_0 < x_1 < \cdots < x_m = b$

$c = y_0 < y_1 < \cdots < y_n = d$

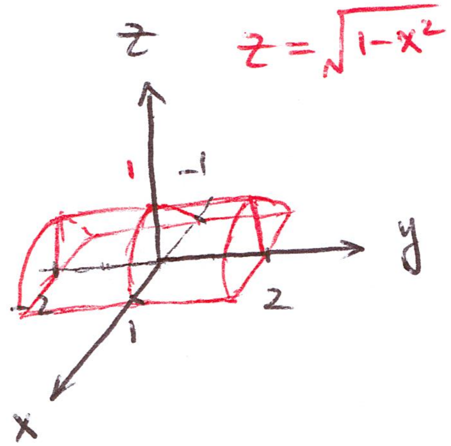
$$V = \iint_R f(x,y)\,dA =$$

Examples

(1) Estimate the volume of the solid that lies above the square $R=[0,2]\times[0,2]$ and below the elliptic paraboloid $z=16-x^2-2y^2$. Divide R into 4 equal squares and choose the sample point to the upper right corner of each square R_{ij}. (see Fig. 8 too)



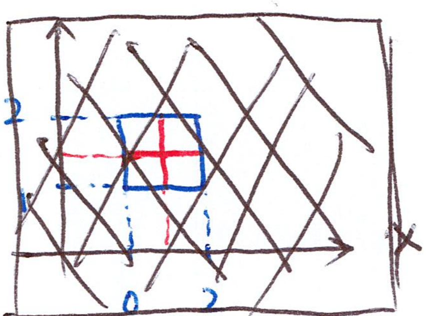
(2) $R=\{(x,y)\mid -1\le x\le 1,\ -2\le y\le 2\}$, evaluate the integral $\iint_R \sqrt{1-x^2}\,dA$.

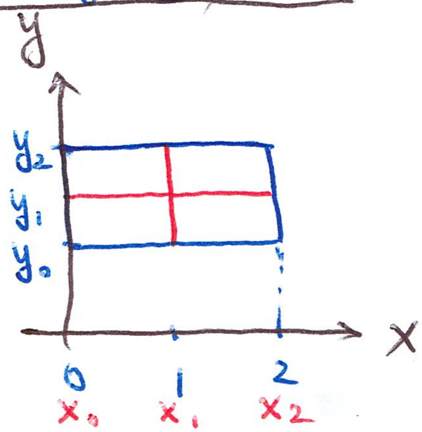


- the midpoint rule

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\,\Delta A, \quad (\bar{x}_i,\bar{y}_j) \text{ is the center of } R_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$$

Ex.3 Use the midpoint rule with $m=n=2$ to estimate the value of the integral $\iint_R (x-3y^2)\,dA$, where $R=\{(x,y)\mid 0\le x\le 2,\ 1\le y\le 2\}$.





§15.2 Iterated Integral

Fubini's Thrm If f is continuous on the rectangle $R=\{(x,y) \mid a\le x\le b,\ c\le y\le d\}$,

$$\iint_R f(x,y)\,dA=\int_a^b\left[\int_c^d f(x,y)\,dy\right]dx=\int_c^d\left[\int_a^b f(x,y)\,dx\right]dy.$$

Remark This is true if f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Examples

(1) $\iint_R (x-3y^2)\,dA$, $R=[0,2]\times[1,2]$

(2) $\iint\limits_{R} y \sin(xy)\, dA$

$R = [1,2] \times [0,\pi]$

(3) Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x=2$ and $y=2$, and the three coordinate planes.

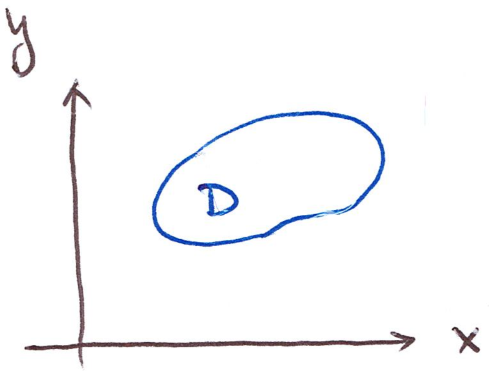
$$f(x,y) = g(x)\,h(y) \implies \iint_R f(x,y)\,dA = \left[\int_a^b g(x)\,dx\right] \cdot \left[\int_c^d h(y)\,dy\right]$$

Ex. 5

$$\iint_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} \sin x \cos y \, dA$$

$$=$$

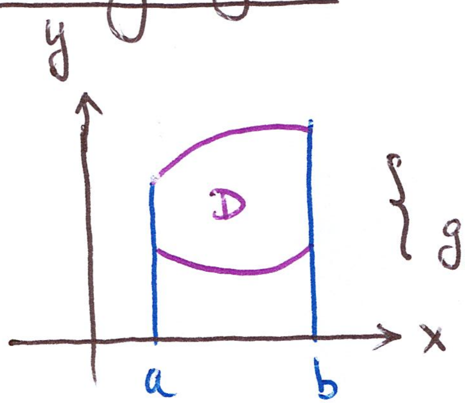
§15.3 Double Integrals over General Regions



general region

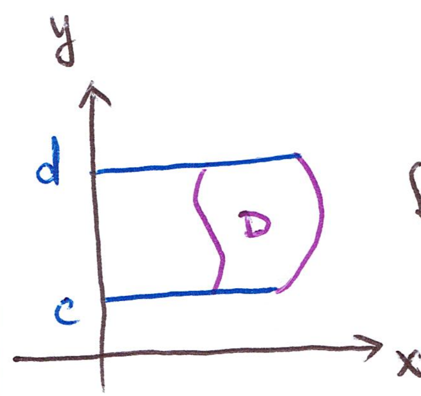
$$\iint_D f(x,y)\,dA =$$

- elementary regions



type - I

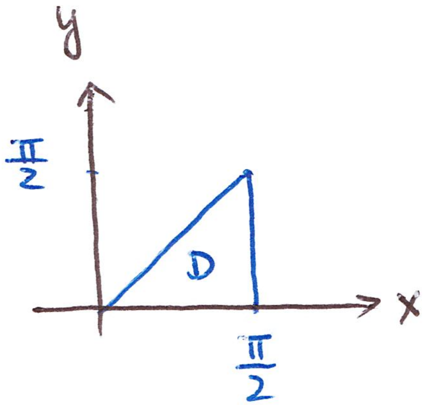
$$\begin{cases} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{cases}$$

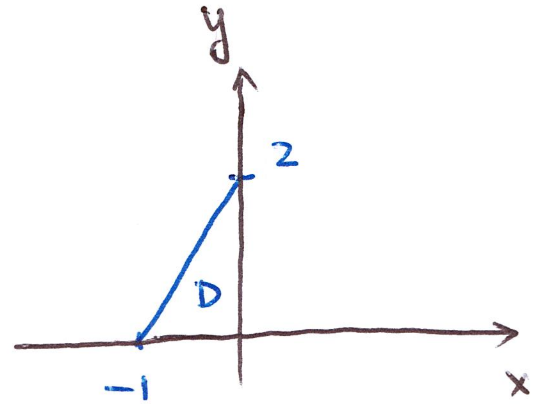


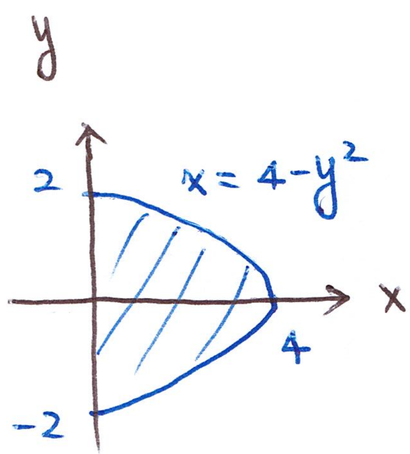
type - II

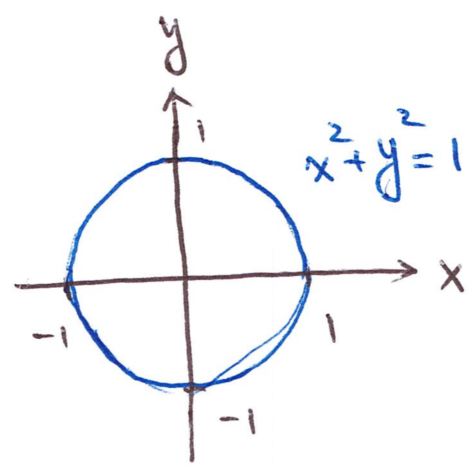
$$\begin{cases} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{cases}$$

<u>examples</u> express the following regions as type-I and/or type II regions









- D is type-I

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

- D is type-II

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

examples

(1) $\iint_D (x+2y)\,dA$, D is the region bounded by the parabolas $y=2x^2$ and $y=1+x^2$

(2) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

(3) $\iint_D xy\, dA$, where D is the region bounded by the line $y = x-1$ and the parabola $y^2 = 2x+6$.

(4) Find the volume of the tetrahedron bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$, and $z=0$.

(4) Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$

- Properties of Double Integrals

(1) $\iint_D (f+g)\,dA = \iint_D f\,dA + \iint_D g\,dA$; (2) $\iint_D c\,f(x,y)\,dA = c\iint_D f\,dA$

(3) $f(x,y) \geq g(x,y)\ \forall (x,y) \in D \implies \iint_D f\,dA \geq \iint_D g\,dA$

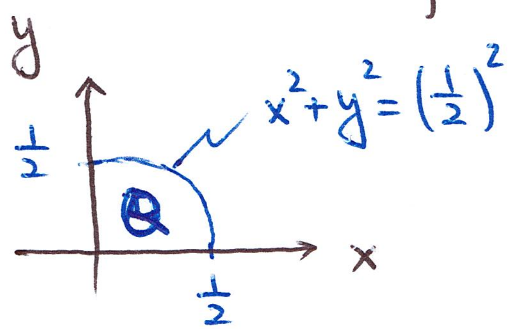
(4) $\iint_{\substack{D=D_1\cup D_2 \\ D_1\cap D_2=\phi}} f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$

(5) $\iint_D 1\,dA = A(D)$

(6) $m \leq f(x,y) \leq M\ \forall (x,y) \in D \implies m \leq \frac{1}{A(D)} \iint_D f\,dA \leq M$.

Examples Use property (6) to estimate the value of the integral.

#57 $\iint_Q e^{-(x^2+y^2)^2}\,dA$,

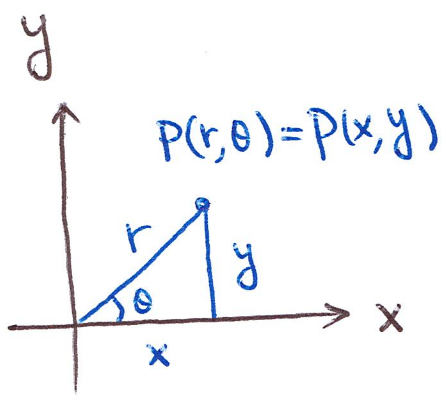


#58 $\iint_T \sin^4(x+y)\,dA$, T is the triangle enclosed by the lines $y=0$, $y=2$, and $x=1$.

§15.4 Double Integrals in Polar Coordinates

- **Polar Coordinates**

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$$



- **change to Polar Coordinates in a double integral**

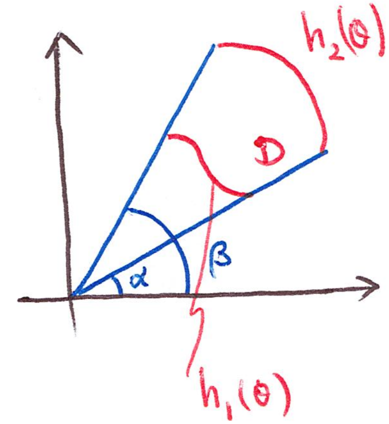
f is continuous on a polar rectangle R: $\begin{matrix} a \le r \le b \\ \alpha \le \theta \le \beta \end{matrix}$, where $0 \le \beta - \alpha \le 2\pi$.

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Ex.1 $\iint\limits_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

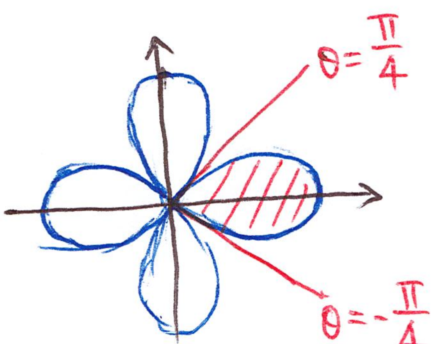
Ex.2 Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.

- $D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$



$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Ex. 3 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.



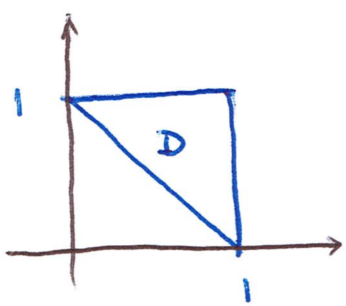
Ex. 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

§15.5 Applications of Double Integrals

- density and mass (the charge density and electric charge)

$$m = \iint_D \rho(x,y)\, dA$$

Ex. 1 Charge is distributed over the region D so that the charge density at (x,y) is $\sigma(x,y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.



- moments and centers of mass

moments

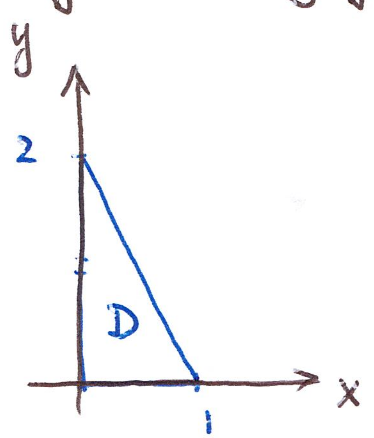
$$M_x = \iint_D y\,\rho(x,y)\,dA\,, \qquad \text{about the x-axis}$$

$$M_y = \iint_D x\,\rho(x,y)\,dA\,, \qquad \text{about the y-axis}$$

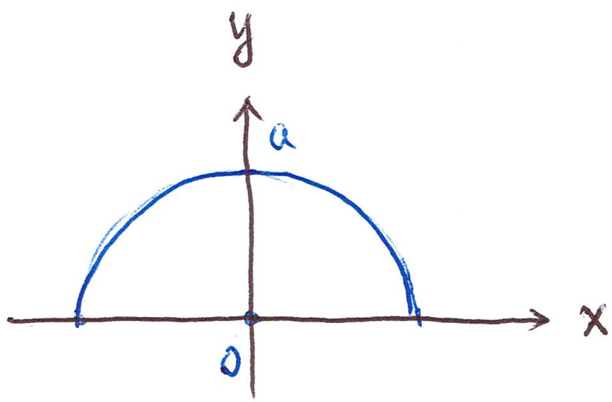
the center of mass

$$(\bar{x}, \bar{y}) = \frac{1}{m}\left(M_y, M_x\right)$$

Ex. 2 Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, and $(0,2)$ if the density function is $\rho(x,y) = 1 + 3x + y$.

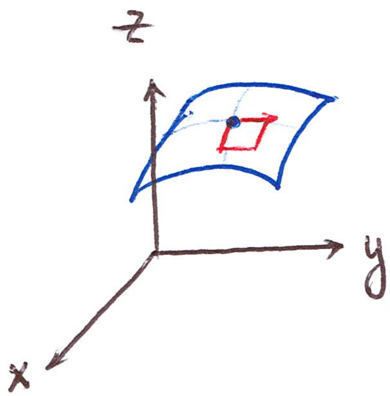


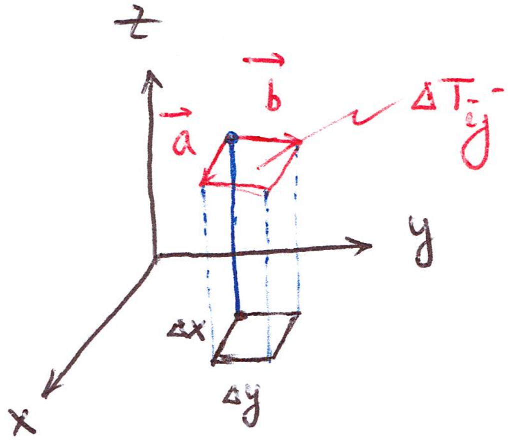
Ex. 3 The density at any point on a ~~semicircle~~ semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.



§15.6 Surface Area

<u>surface S</u> : $z = f(x,y)$, $(x,y) \in D$





$$\vec{a} = \langle \Delta x,\ 0,\ \frac{\partial f}{\partial x}(x_i, y_j)\,\Delta x \rangle$$

$$\vec{b} = \langle 0,\ \Delta y,\ \frac{\partial f}{\partial y}(x_i, y_j)\,\Delta y \rangle$$

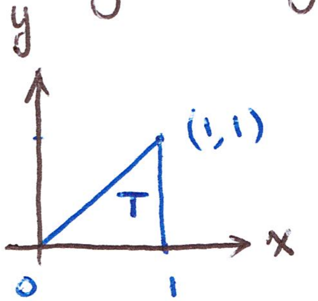
$$\Delta T_{ij} = |\vec{a} \times \vec{b}|$$

$$= \sqrt{1 + \frac{\partial f}{\partial x}(x_i, y_j)^2 + \frac{\partial f}{\partial y}(x_i, y_j)^2}\ \Delta x \Delta y$$

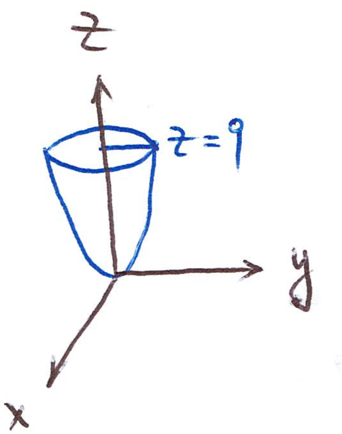
$$A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

$$= \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ dA$$

Ex. 1 Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vercies $(0,0)$, $(1,0)$, and $(1,1)$.



Ex. 2 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.



§15.7 Triple Integrals

Def.

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$

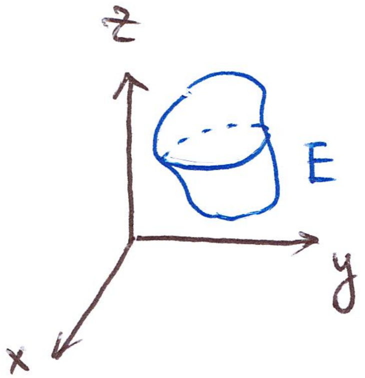
where $B = [a,b]\times[c,d]\times[r,s]$

Fubini's Thrm f is continuous on the rectangular box $B = [a,b]\times[c,d]\times[r,s]$

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Ex. 1 $\iiint_B xyz^2\,dV$, $B = [0,1]\times[-1,2]\times[0,3]$

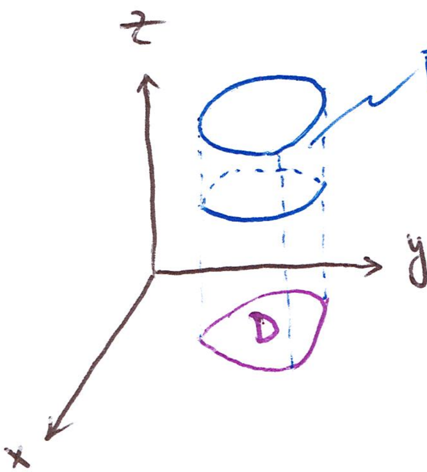
- <u>general region</u>



$$\iiint_E f(x,y,z)\,dV =$$

- <u>elementary regions</u>

$$\iiint_E f\,dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f\,dz\,dA$$



$$E : \begin{cases} u_1(x,y) \le z \le u_2(x,y) \\ (x,y) \in D \end{cases}$$

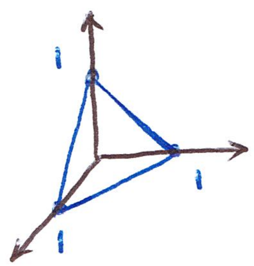
type-I

type-II $\begin{cases} u_1(x,z) \le y \le u_2(x,z) \\ (x,z) \in D \end{cases}$

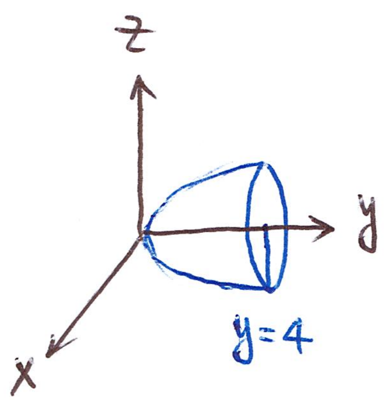
type-III $\begin{cases} u_1(y,z) \le x \le u_2(y,z) \\ (y,z) \in D \end{cases}$

examples

(1) Evaluate $\iiint_E z \, dV$, where E is the solid tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$, and $x+y+z=1$.



(2) Evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$, where E is the region bounded by the paraboloid $y=x^2+z^2$ and the plane $y=4$



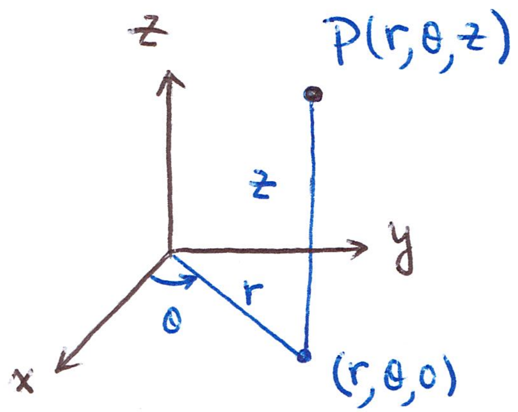
(3) Express the iterate integral $\int_0^1 \int_0^{x^2} \int_0^{y} f(x,y,z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterate integral in a different order, integrating first with respect to x, then z, and then y.

(4) Find the volume of the solid enclosed by the paraboloid $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.

#16 (P1038) $\iiint_T xz\,dV$, where T is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$

#18 (P1038) $\iiint_E z \, dV$, where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant.

§15.7 Triple Integrals in Cylindrical Coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \\ z = z \end{cases}$$

Ex. 1 (a) Plot the point with cylindrical coordinates $\left(2, \frac{2\pi}{3}, 1\right)$ and find its rectangular coordinates.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Ex. 2 Describe the surface whose equation in cylindrical coordinates is $z = r$.

write the equations in cylindrical coordinates

#9 (a) $x^2 - x + y^2 + z^2 = 1$

#9 (b) $z = x^2 - y^2$

Sketch the solid described by the given inequalities

#11 $r^2 \leq z \leq 8 - r^2$

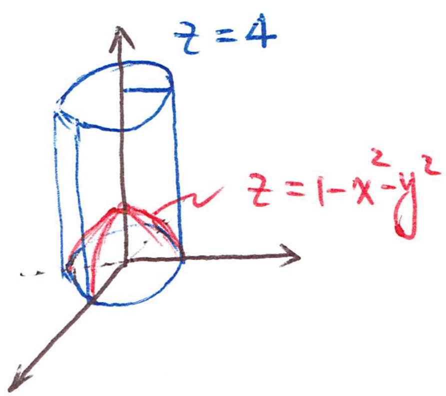
#12 $0 \leq \theta \leq \frac{\pi}{2}$, $r \leq z \leq 2$

$$E = \{ (x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y) \}$$

$$D = \{ (r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta) \}$$

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

<u>Ex. 3</u> A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

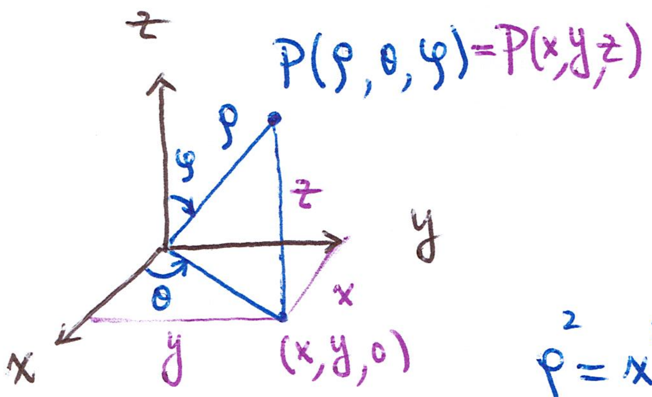


Ex. 4 $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$ using the cylindrical coordinates

#22 Find the volume of the solid that lies within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.

§15.9 Triple Integrals in Spherical Coordinates

- spherical coordinates



$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$\rho \geq 0$

$0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$

$$\rho^2 = x^2 + y^2 + z^2$$

Surfaces (a) $\rho = c$; (b) $\theta = c$; (c) $\varphi = c$. c — constant

Ex. 1 Find its rectangular coordinates of the point $(2, \frac{\pi}{4}, \frac{\pi}{3})$ given in spherical coordinates.

Ex. 2 Find the spherical coordinates of the point $(0, 2\sqrt{3}, -2)$ given in rectangular coordinates.

- triple integral in spherical coordinates

$$\iiint_E f(x,y,z)\, dV = \iiint_E f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \rho^2\sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| =$$

Ex. 3 $\displaystyle\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where B is the unit ball, $B=\{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$

Ex. 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and ~~above~~ below the sphere $x^2+y^2+z^2=z$.

#40 Evaluate $\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}(x^2z+y^2z+z^3)\,dV$ by using spherical coordinates.

#25 $\iiint_E x e^{x^2+y^2+z^2}\,dV$, E: the portion of the unit ball $x^2+y^2+z^2\le 1$ that lies in the 1st octant.