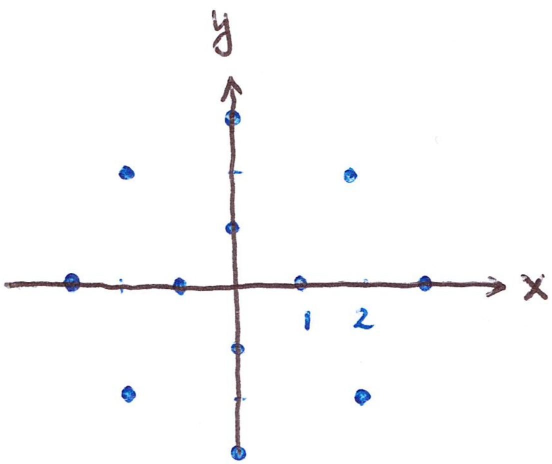
Chapter 16 Vector Calculus (12 lectures)

§16.1 Vecto Fields

vector fields $\quad \vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} = \langle P(x,y), Q(x,y)\rangle : R^2 \to R^2$

$$\vec{F}(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle : R^3 \to R^3$$

Ex. 1 Describe $\vec{F}(x,y) = \langle -y, x\rangle$ by sketching some of the vectors $\vec{F}(x,y)$ as in Fig. 3.



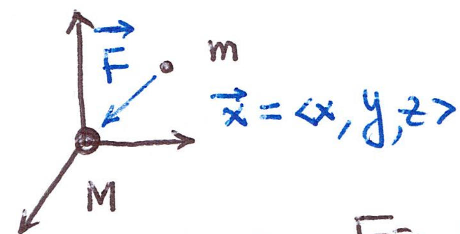
$$\vec{x}\cdot\vec{F}(\vec{x}) =$$

$$|\vec{F}(x,y)| =$$

Ex. 2 Sketch the vector field on R^3 given by $\vec{F}(x,y,z) = \langle 0, 0, z \rangle$.

Ex. 4 Gravitational Force

$$\vec{F}(\vec{x}) = -\frac{mMG}{|\vec{x}|^3}\vec{x}$$



see Fig. 14

$$= \nabla f, \quad \text{where } f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$$

- <u>gradient fields</u>

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

<u>Ex. 6</u> Find the gradient vector field of $f(x,y) = x^2y - y^3$.
Plot the gradient field together with a contour map of f (see Fig. 15).
How are they related?

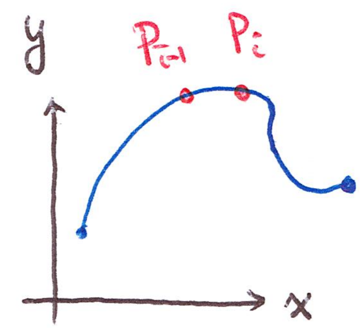
§16.2 Line Integrals

$\left\langle \begin{array}{l} \text{scalar function} \\ \text{vector field} \end{array}\right.$

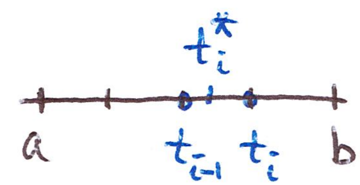
- <u>line integral of scalar function</u>

<u>a plane curve</u> $C : \vec{r}(t) = \langle x(t), y(t) \rangle$

$a \le t \le b$



Assume that C is a smooth curve. ($\vec{r}\,'$ is cont. and $\vec{r}\,'(t) \neq \vec{0}$)



Partition of $[a,b]$: $a = t_0 < t_1 < t_2 \cdots < t_n = b$

$t_i = a + i\Delta t, \quad \Delta t = \frac{b-a}{n}$

<u>definition</u> the line integral of f along C

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

$$= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

Examples

(1) Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper half of the unit circle $x^2+y^2=1$

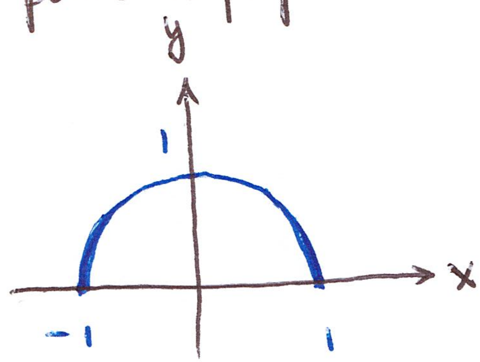
(2) Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

the wire with density function ρ

mass $\quad m = \int_C \rho \, ds$

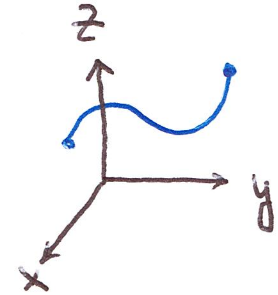
center of mass $\quad (\bar{x}, \bar{y}) = \left(\frac{1}{m} \int_C x \rho(x,y) \, ds, \; \frac{1}{m} \int_C y \rho(x,y) \, ds \right)$

Ex. 3 A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geq 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.



a space curve $C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$a \le t \le b$$

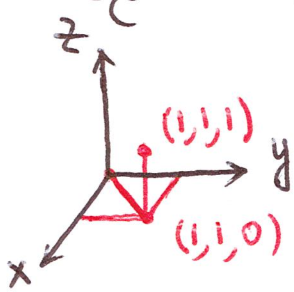


$$\int_C f(x,y,z)\,ds = \int_a^b f\big(x(t), y(t), z(t)\big) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_a^b f\big(\vec{r}(t)\big)\, \big|\vec{r}\,'(t)\big|\, dt$$

Ex. 5 Evaluate $\int_C y \sin z\, ds$, where C is the circular helix given by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$.

Ex. 6 $\int_C (x - 3y^2 + z)\,ds$



C: line segments
from (0,0,0) to (1,1,0)
and from (1,1,0) to (1,1,1)

- <u>line integral of vector field</u>

<u>vector field</u> $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

<u>space curve</u> C : $\boxed{\cancel{\vec{r}(x,y,z)=t}}$ $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$

the line integral of $\vec{F}$ along C

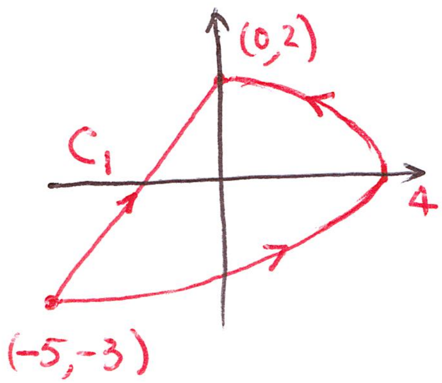
$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left(\vec{F} \cdot \vec{T} \right) ds$$

where $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ – is the unit tangent vector

$$=$$

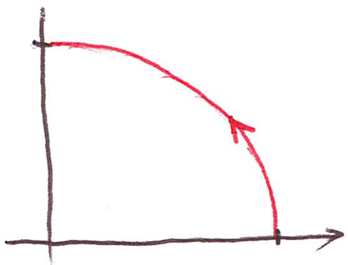
Examples

(1) Evaluate $\int_C y^2 dx + x\,dy$, where $C = C_1 \cup C_2$, C_1 is the line segment from $(-5,-3)$ to $(0,2)$ and C_2 is the arc of $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.



(2) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle xy, yz, zx \rangle$ and C is parametrized by $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$

(3) Find the work done by the force field $\vec{F}(x,y) = \langle x^2, -xy \rangle$ in moving a particle along the quarter-circle $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq \frac{\pi}{2}$.



§16.3 The Fundamental Theorem for Line Integrals

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\parallel$$

$$\int_C \langle F', 0\rangle \cdot \vec{r}\,' dt = \int_C \langle F', 0\rangle \cdot d\vec{r}$$

C: line segment $a \to b$

$\vec{r}(t) = \langle t, 0\rangle$

$a \le t \le b$

<u>Theorem</u> Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$.
Let f be a differentiable function whose gradient vector ∇f is continuous on C.

$$\int_C \nabla f \cdot d\vec{r} = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right).$$

<u>Proof</u> $\displaystyle\int_C \nabla f \cdot d\vec{r} =$

Ex. 1 Find the work done by the gravitational field $\vec{F}(\vec{x}) = -\dfrac{mMG}{|\vec{x}|^3}\vec{x}$ in moving a particle with mass m from the point $(3,4,12)$ to the point $(2,2,0)$ along a piecewise smooth curve.

Theorem Assume that $\vec{F} \in C^1(R^3)^3$ and that C is a piecewise smooth curve. The following statements are equivalent.

(1) $\forall$ simple close curve C: $\int_C \vec{F}\cdot d\vec{r} = 0$.

(2) $\int_C \vec{F}\cdot d\vec{r}$ is independent of path.

(3) there exists a function f such that $\vec{F} = \nabla f$.

(4) $\nabla\times\vec{F} = \vec{0}$ on a simply connected domain.

$\nabla\times\vec{F} =$

<u>Proof</u> $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$

- <u>curl of vector field</u>

$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$; $\nabla \times \vec{F} =$

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$, $\nabla \times \vec{F} =$

<u>Examples</u>

(1) Determine whether or not the vector field $\vec{F}(x,y) = \langle x-y, x-2 \rangle$ is conservative.

and $\vec{F}(x,y) = \langle 3+2xy, x^2-3y^2 \rangle$

If it is conservative, find f such that $\vec{F} = \nabla f$, and evaluate $\int_C \vec{F} \cdot d\vec{r}$,

where C is a curve given by $\vec{r}(t) = \langle e^t \sin t, e^t \cos t \rangle$, $0 \le t \le \pi$.

(2) Is $\vec{F} = \langle y^2, 2xy + e^{3z}, 3y e^{3z} \rangle$ conservative?

If so, find f such that $\vec{F} = \nabla f$.

- <u>conservation of energy</u>

A continuous force field $\overset{=\nabla f}{\vec{F}}$ moves an object along a path $C: \vec{r}(t)$, $a \le t \le b$, where $\vec{r}(a) = A$ is the initial point and $\vec{r}(b) = B$ is the terminal point of C.

$$\underbrace{P(A)}_{\text{potential energy}} + \underbrace{K(A)}_{\text{kinetic energy}} = P(B) + K(B)$$

where $P = -f$ and $K(A) = \frac{1}{2} m |\vec{v}(a)|^2$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt =$$

$$= -\int_C \nabla P \cdot d\vec{r} =$$

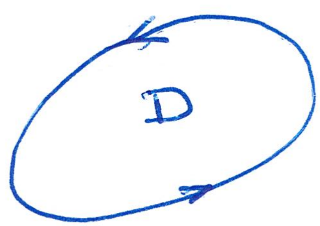
§16.4 Green's Theorem (2 dimensions)

vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

divergence $\nabla \cdot \vec{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$

curl $\nabla \times \vec{F} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then



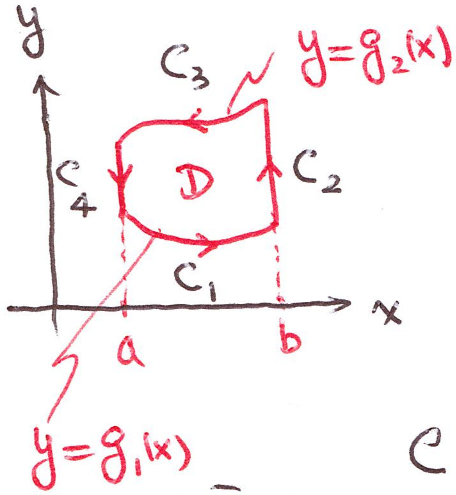
$C = \partial D^{+}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D \nabla \times \vec{F}\, dA$$

$$\int_C (\vec{F} \cdot \vec{n})\, ds = \int_C P\,dy - Q\,dx = \iint_D \nabla \cdot \vec{F}\, dA$$

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

Proof of $\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$



$$\iint_D \frac{\partial P}{\partial y}\,dA =$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

parametrizations of curves

C_1 :

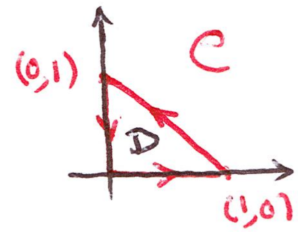
C_2 :

C_3 :

C_4 :

Examples

(1) Evaluate $\int_C x^4\,dx + xy\,dy$, where



(2) Evaluate $\oint_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$, where C is the circle $x^2+y^2=9$.

- Area of D

$$A(D) = \iint_D \frac{\partial x}{\partial x}\, dA =$$

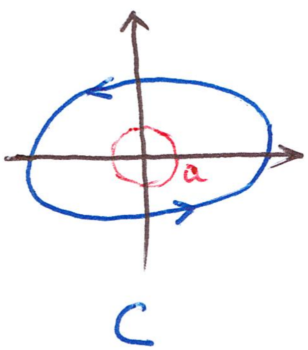
$$= \iint_D \frac{\partial y}{\partial y}\, dA =$$

Ex. 3 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ex. 4 Evaluate $\int_C y^2 dx + 3xy\, dy$, where C is the boundary of the semiannular region D in the upper half plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Ex. 5 $\vec{F}(x,y) = \left\langle -\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$,

Show that $\int_C \vec{F} \cdot d\vec{r} = 2\pi$ for every positively oriented simple closed path that encloses the origion.



§16.5 Curl and Divergence

- $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

- curl

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

Ex. 1 $\vec{F} = \langle xz, xyz, -y^2 \rangle$, $\nabla \times \vec{F} =$

<u>Theorem</u> Assume that $f(x,y,z)$ has continuous second-order partial derivatives

$$\Longrightarrow \quad \text{curl}\,(\nabla f) = \vec{0}$$

- $\vec{F}$ is conservative $\Longleftrightarrow$ $\vec{F} = \nabla f$.

<u>Ex. 2</u> Show that the vector field $\vec{F} = \langle xz,\ xyz,\ -y^2 \rangle$ is not conservative.

<u>Theorem</u> Assume that $\vec{F}(x,y,z)$ has continuous partial derivatives and $\text{curl}\,\vec{F} = \vec{0}$

$$\Longrightarrow \quad \vec{F} = \nabla f.$$

Ex. 3 (a) Show that $\vec{F} = \langle y^2 z^3, 2xyz^3, 3xy^2z^2 \rangle$ is a conservative vector field.

(b) Find a function f such that $\vec{F} = \nabla f$.

- divergence

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

Ex. 4 $\vec{F} = \langle xz, xyz, -y^2 \rangle$

$\operatorname{div} \vec{F} =$

Theorem $\vec{F} = \langle P, Q, R \rangle$, P, Q, R have continuous second-order partial derivatives

$$\Longrightarrow \operatorname{div} \operatorname{curl} \vec{F} = 0$$

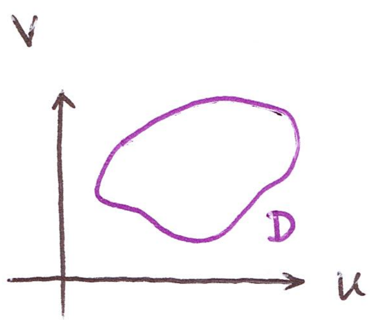
Proof

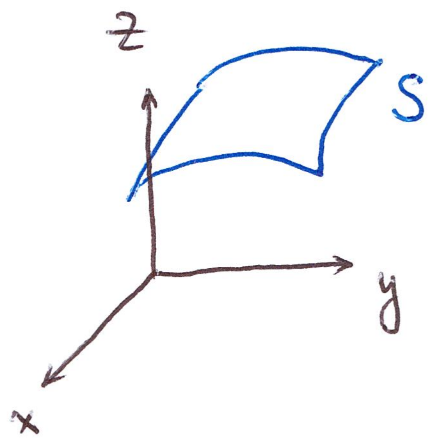
Ex. 5 Show that $\vec{F} = \langle xz, xyz, -y^2 \rangle$ cannot be written as the curl of another vector field, that is, $\vec{F} \neq \operatorname{curl} \vec{G}$

§16.6 Parametric Surfaces and Their Areas

curves in R^2 $\begin{cases} \text{graph} \\ \text{level curve} \\ \text{parametric curve} \end{cases}$

surfaces in R^3 $\begin{cases} \text{graph} \\ \text{level surface} \\ \text{parametric surface} \end{cases}$





Ex. 1 Identify the sketch the surface with vector equation

$$\vec{r}(u,v) = \langle 2\cos u,\ v,\ 2\sin u \rangle$$

Examples Find parametric representations.

(1) the plane passing through three points (x_0, y_0), (x_1, y_1), and (x_2, y_2)

(2) the sphere $x^2 + y^2 + z^2 = a^2$

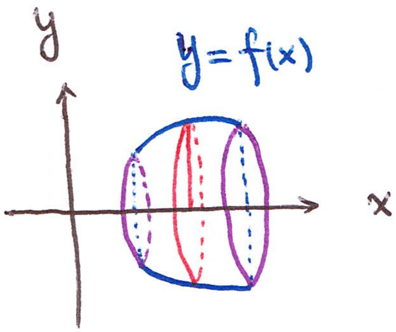
(3) the cylinder $x^2+y^2=4$, $0 \le z \le 1$.

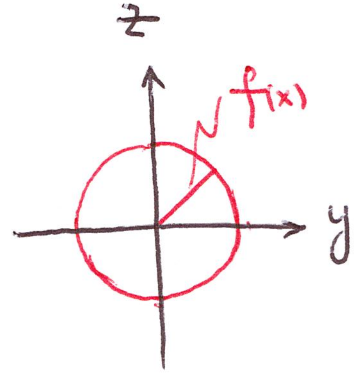
(4) the elliptic paraboloid $z = x^2+2y^2$.

(5) the surface $z = 2\sqrt{x^2+y^2}$, that is, the top half of the cone $z^2=4x^2+4y^2$.

(6) the graph $z=f(x,y)$, $(x,y)\in D$.

- <u>surfaces of revolution</u>





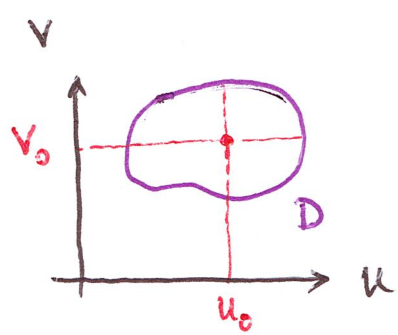
parametrization

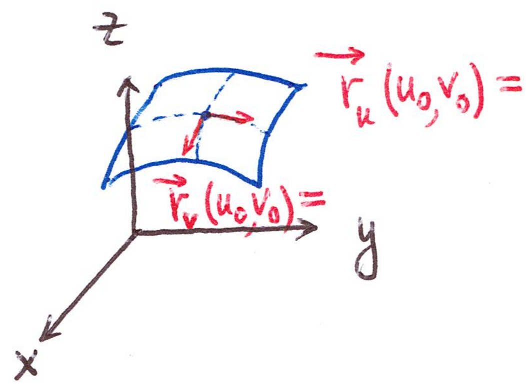
$$\begin{cases} x = \\ y = \\ z = \end{cases}$$

<u>Ex. 8</u> Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \leq x \leq 2\pi$, about the x-axis.

- <u>surface</u> $S = \{ \vec{r}(u,v) \mid (u,v) \in D \}$

(1) S is smooth at $\vec{r}(u_0, v_0) \iff \vec{r}_u \times \vec{r}_v (u_0, v_0) \neq 0$





(2) <u>tangent plane at $\vec{r}(u_0, v_0)$</u> to surface $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

- <u>parametric representation</u>

- <u>graph representation</u>

<u>Ex. 9</u> Find the tangent plane to the surface $\vec{r}(u,v) = \langle u^2, v^2, u+2v \rangle$ at $(1,1,3)$.

(3) surface area

- parametric surface $S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, $(u,v) \in D$

is a smooth surface and is covered once

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA \qquad dS = |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

Ex. 10 Find the surface area of a sphere of radius a.

• <u>graph</u> $S: z = f(x,y), (x,y) \in D$

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

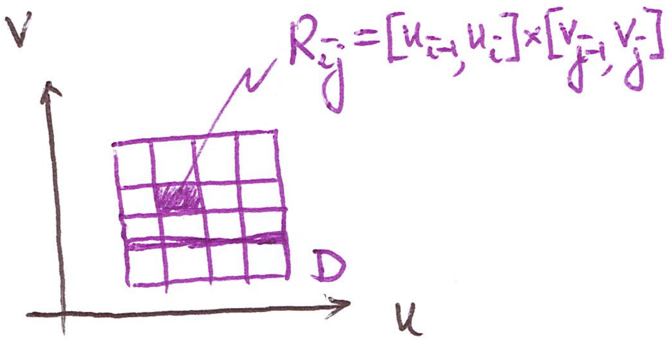
<u>Ex. 11</u> Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

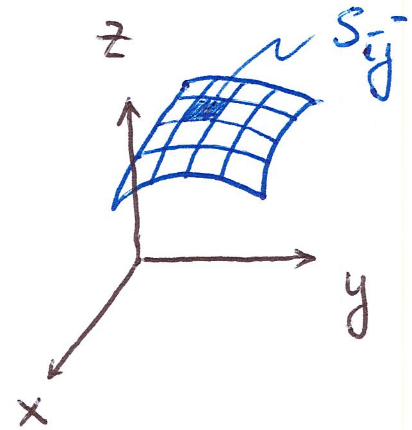
§16.7 Surface Integrals

- scalar function
- vector field

- surface integrals of scalar functions

(a) parametrized surface $S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \; (u,v) \in D$





$P^*_{ij} \in S_{ij}$

$$\iint_S f(x,y,z)\, dS = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P^*_{ij})\, \Delta S_{ij}$$

$$=$$

$A(S) =$

Ex. 1 Compute the surface integral $\iint_S x^2 \, dS$, S is the unit sphere $x^2+y^2+z^2=1$.

- center of mass ρ is the density

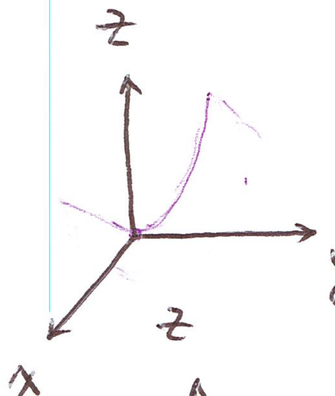
mass $m = \iint_S \rho \, dS$

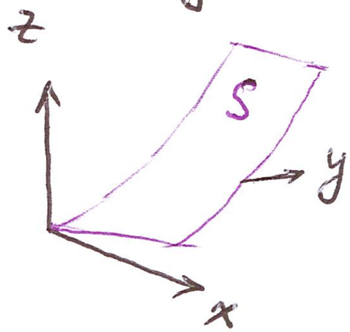
center of mass $(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{m}\left(\iint_S x\rho \, dS, \iint_S y\rho \, dS, \iint_S z\rho \, dS \right)$

(b) <u>graph of function</u> $S: z = f(x,y),\ (x,y)\in D$

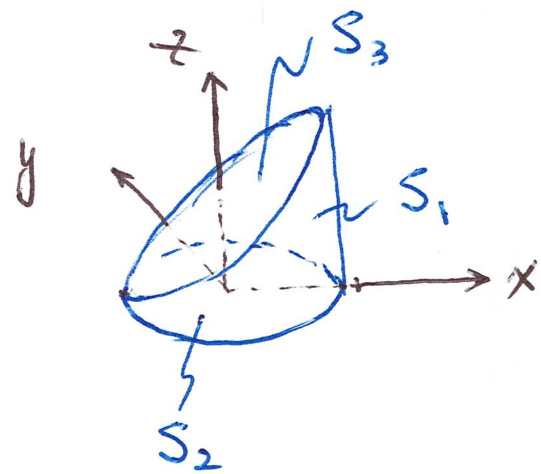
$$\iint_S f(x,y,z)\,dS = \iint_D$$

<u>Ex. 2</u> Evaluate $\iint_S y\,dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$.



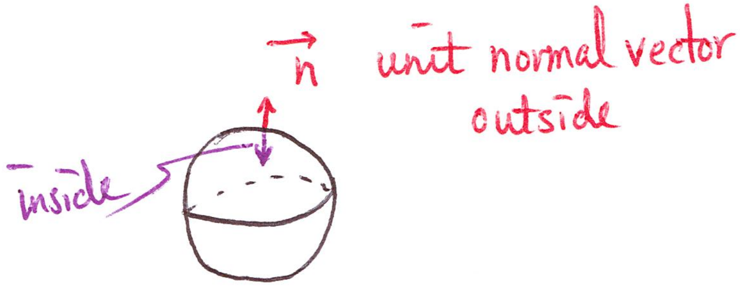


Ex. 3 Evaluate $\iint_S z \, dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2+y^2=1$, whose bottom S_2 is the disk $x^2+y^2 \leq 1$ in the plane $z=0$, and whose top S_3 is the part of the plane $z=1+x$ that lies above S_2.

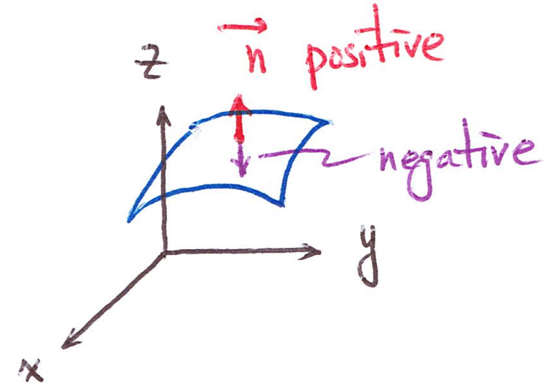


- <u>surface integral of vector fields</u>

(a) oriented surface $S \iff S$ has two sides (positive/neg. or outside/inside).



closed surface



<u>sphere</u> $x^2+y^2+z^2=a^2$

$$\vec{r}(\theta,\varphi)\ \begin{cases} x = a\cos\theta\sin\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\varphi \end{cases} \qquad \begin{matrix} 0\le\theta\le 2\pi \\ 0\le\varphi\le\pi \end{matrix}$$

$$\vec{r}_\varphi \times \vec{r}_\theta = a\sin\varphi\ \underline{\vec{r}(\theta,\varphi)}$$ <u>outside</u>

<u>graph</u> $z=f(x,y)$, $(x,y)\in D$

$$\begin{cases} x = x \\ y = y \\ z = f(x,y) \end{cases}$$

$$\vec{r}_x \times \vec{r}_y = \langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, \underline{1} \rangle$$ <u>positive</u>

(b) <u>parametric surface</u> $S : \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, $(u,v) \in D$.
is oriented with unit normal vector

the flux of $\vec{F}$ across S

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \left(\vec{F} \cdot \vec{n}\right) dS$$

$$= \iint_D$$

<u>Ex. 4</u> Find the flux of $\vec{F}(x,y,z) = \langle z, y, x \rangle$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

(c) graph $S: z = g(x,y),\ (x,y) \in D$ is oriented with upward orientation.

$$\iint_S \vec{F} \cdot d\vec{S} =$$

Ex. 5 Evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = \langle y, x, z \rangle$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

(d) heat flow $\vec{F} = -K \nabla u$

conductivity

temperature

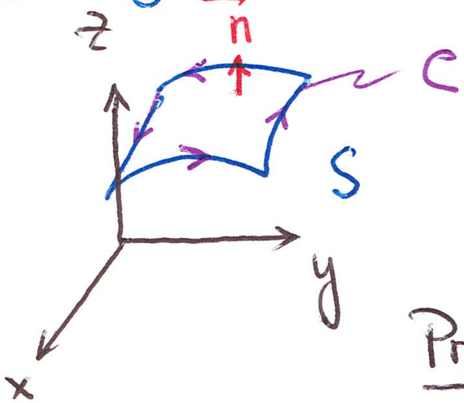
the rate of heat flow across the surface

$$\iint_S \vec{F} \cdot d\vec{S} = -K \iint_S \nabla u \cdot d\vec{S}$$

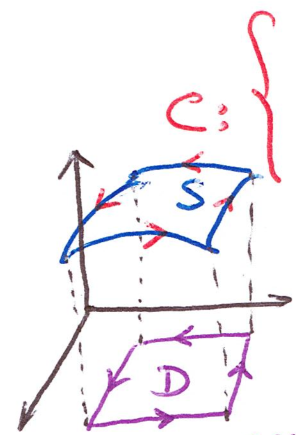
Ex. 6 The temperature u in a metall ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

§16.8 Stoke's Theorem

Stoke's Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains S.



$$\int_{C=\partial S} \vec{F}\cdot d\vec{r} = \iint_S \overrightarrow{\nabla\times F}\cdot d\vec{S}$$



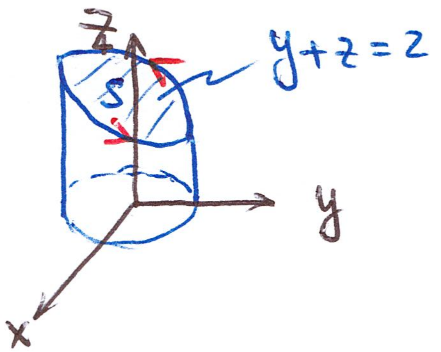
$C_1: \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad a \le t \le b$

Proof of S: $z = g(x,y)$, $(x,y)\in D$

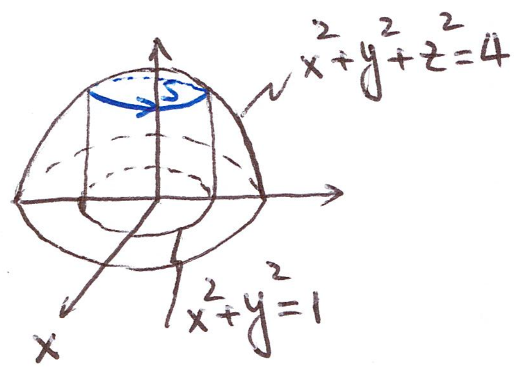
$$\iint_S \overrightarrow{\nabla\times F}\cdot d\vec{S} =$$

$$\int_C \vec{F}\cdot d\vec{r} =$$

Ex. 1 Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.



Ex. 2 Use Stoke's ~~Theorem~~ Theorem to compute $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$, where $\vec{F} = \langle xz, yz, xy \rangle$ and S is the part of $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.



• S_1 and S_2 are surfaces having the same orientation, and $\partial S_1 = \partial S_2 = C$

$$\iint_{S_1} \nabla \times \vec{F} \cdot d\vec{S} \qquad\qquad \iint_{S_2} \nabla \times \vec{F} \cdot d\vec{S}$$

Examples Use Stoke's Theorem to evaluate $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$

#2 (P1139) $\vec{F} = \langle x^2 \sin z, y^2, xy \rangle$, S is the part of $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward.

#6 (P1139) $\vec{F} = \langle e^{xy}, e^{xz}, x^2 z \rangle$, S is the half of $4x^2 + y^2 + 4z^2 = 4$ that lies to the right of the xz-plane, oriented in the direction of the positive y-axis.

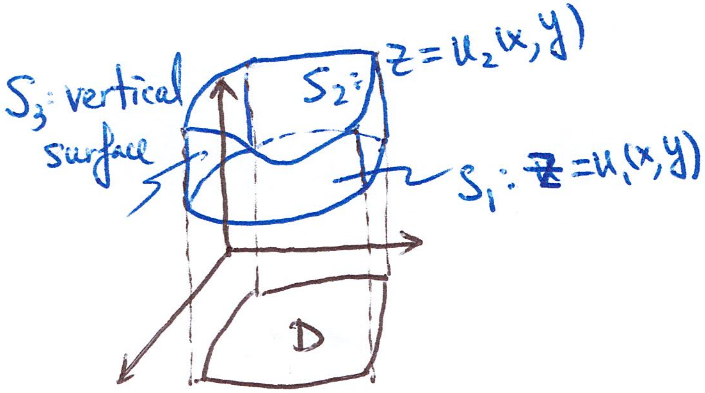
§16.9 The Divergence Theorem

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region that contains E.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (\nabla\cdot\vec{F})\, dV.$$

$$\vec{F} = \langle P, Q, R \rangle$$

Proof of $\displaystyle\iint_S R(\vec{k}\cdot\vec{n})\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV$

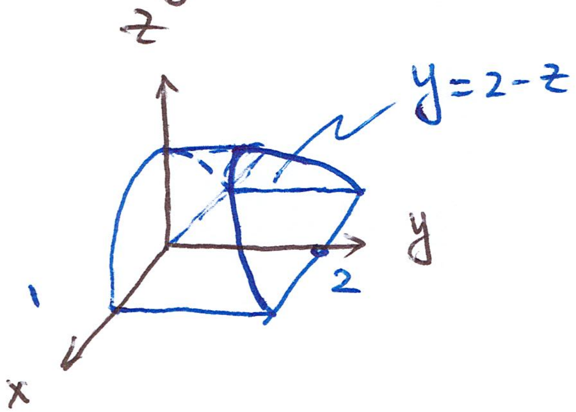


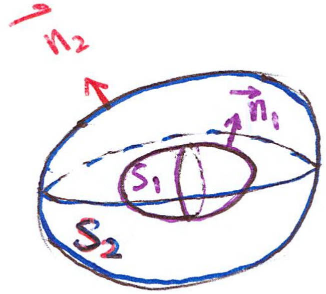
$E: u_1(x,y) \le z \le u_2(x,y)$

$(x,y)\in D$

Ex. 1 Find the flux of $\vec{F} = \langle z, y, x \rangle$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Ex. 2 Evaluate $\iint_S \vec{F} \cdot d\vec{S}$, $\vec{F} = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle$, S is the surface of the region E bounded by $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.





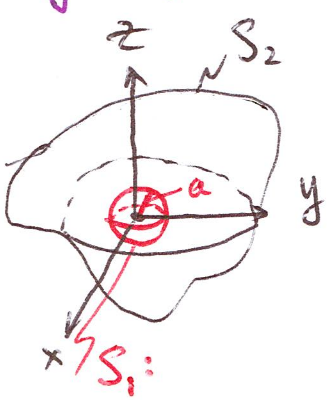
The region E lies between the closed surfaces S_1 and S_2.

$\partial E = S = S_1 \cup S_2$.

$$\iiint_E \operatorname{div} \vec{F}\, dV = -\iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$$

<u>Ex. 3</u> The electric field $\vec{E}(\vec{x}) = \frac{\varepsilon Q}{|\vec{x}|^3} \vec{x}$, the electric charge Q is located at the origin, $\vec{x} = \langle x, y, z \rangle$ is a position vector. Prove that

the electric flux of $\vec{E}$ $\quad \iint_{S_2} \vec{E} \cdot d\vec{S} = 4\pi\varepsilon Q$, S_2 is any closed surface containing the origin.



<u>Proof</u>

Evaluate $\displaystyle\iint_S \vec{F}\cdot d\vec{S}$

<u>#6 (P1145)</u> $\vec{F} = \langle x^2yz,\ xy^2z,\ xyz^2 \rangle$, S is the surface of the box enclosed by the planes $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, and $z=c$. $(a>0,\ b>0,\ c>0)$.

<u>#10 (P1145)</u> $\vec{F} = \langle z,\ y,\ zx \rangle$, S is the surface of the tetrahedron enclosed by the coordinate planes and the plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$. $(a>0,\ b>0,\ c>0)$.