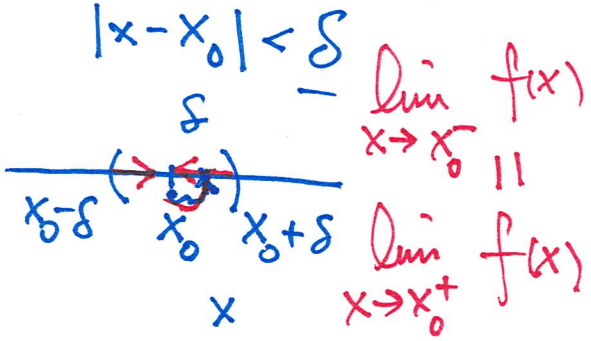
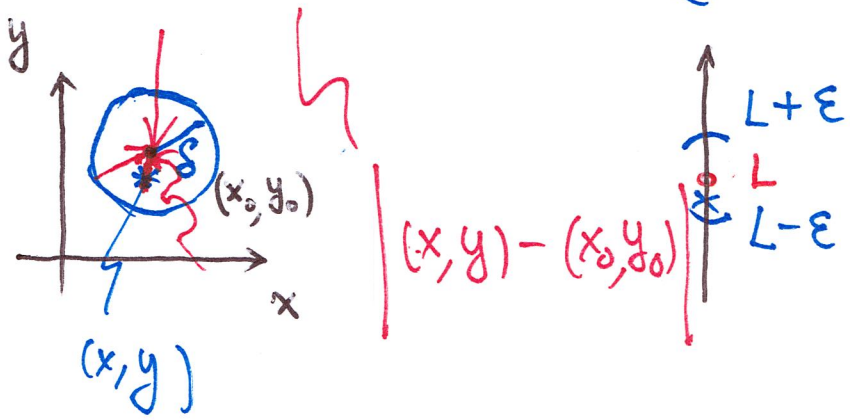
§14.2 Limits and Continuity

- **limits**

$\forall\, \varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$ such that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \iff 0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon$$





$-\delta < x - x_0 < \delta \iff -\delta + x_0 < x < \delta + x_0$

$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ DNE $\iff$ $\lim_{(x,y)\to(x_0,y_0)} f(x,y) \neq \lim_{(x,y)\to(x_0,y_0)} f(x,y)$

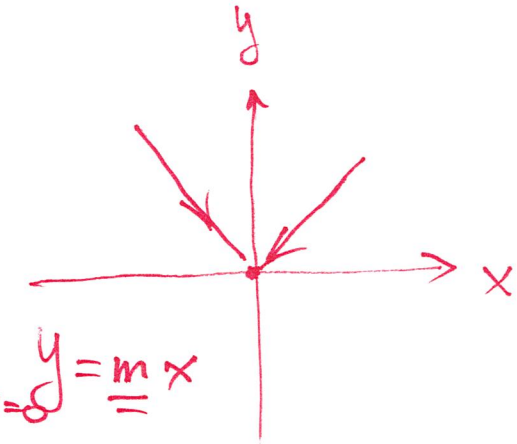
along path 1 along path 2

Ex. 1 Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ DNE.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim \frac{x^2-(mx)^2}{x^2+(mx)^2} = \lim_{(x,y)\to(0,0)} \frac{1-m^2}{1+m^2}$$

$y=mx$

$$= \frac{1-m^2}{1+m^2} = \begin{cases} 1, & m=0 \\ 0, & m=1 \end{cases}$$



examples Do the following limits exist?

(1) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$,

$y=mx$

$$\lim_{(x,y)\to(0,0)} \frac{mx^2}{x^2+m^2x^2}$$

$$= \frac{m}{1+m^2} = \begin{cases} 0, & m=0 \\ \frac{1}{2}, & m=1 \end{cases}$$

(2) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$,

$y=mx$

$$\lim_{(x,y)\to(0,0)} \frac{xm^2x^2}{x^2+m^4x^4} = \lim_{(x,y)\to(0,0)} \frac{m^2x}{1+m^4x^2} = 0$$

$$\lim \frac{xy^2}{x^2+y^4} \quad \overset{y^2=x}{=\!=} \quad \lim \frac{y^2\cdot y^2}{y^4+y^4} = \frac{1}{2}$$

(3) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

$y=mx$

$$\lim \frac{3x^2\cdot mx}{x^2+m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} x\cdot\left(\frac{3m}{1+m^2}\right) = 0$$

$y=x^2$

$$\lim \frac{3x^2\cdot x^2}{x^2+x^4}$$

$$= \lim \frac{3x^2}{1+x^2} = 0$$

$$\left|\frac{3x^2y}{x^2+y^2}\right| \overset{?}{<} \varepsilon$$

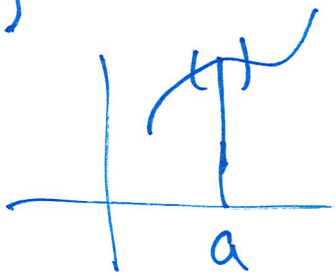
$$\sqrt{y^2} \leq \sqrt{x^2+y^2} < \delta \overset{?}{=}$$

$$3|y|\frac{x^2}{x^2+y^2} \leq 3|y| \leq 3\sqrt{x^2+y^2} < \varepsilon$$

$$\sqrt{x^2+y^2} < \frac{\varepsilon}{3} = \delta$$

- **continuity** $f(x)$ is cont. at $a \iff \lim_{x\to a} f(x) = f(a)$

$f(x,y)$ is continuous at $(a,b) \iff \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$



- polynomials and rational functions are continuous on their domain.

ex.5 $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2\cdot 2^3 - 1^3\cdot 2^2 + 3\cdot 1 + 2\cdot 2 =$

ex.6 Where is the function $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ continuous?

$x^2+y^2 = 0 \Rightarrow (x,y) = (0,0)$

ex.7 Is $g(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$ continuous at $(0,0)$?

$\lim_{(x,y)\to(0,0)} \dfrac{x^2-m^2x^2}{x^2+m^2x^2} = \dfrac{1-m^2}{1+m^2}$

$y = mx$

Ex. 8 Is $f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$ continuous at $(0,0)$?

$\lim_{(x,y)\to(0,0)} f = 0$

Ex. 9 Where is $h(x,y) = \arctan\left(\frac{y}{x}\right)$ continuous?

- **Functions of three or more variables**

Limit $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$

Continuity $f(x,y,z)$ is cont. at $(x_0,y_0,z_0) \iff$