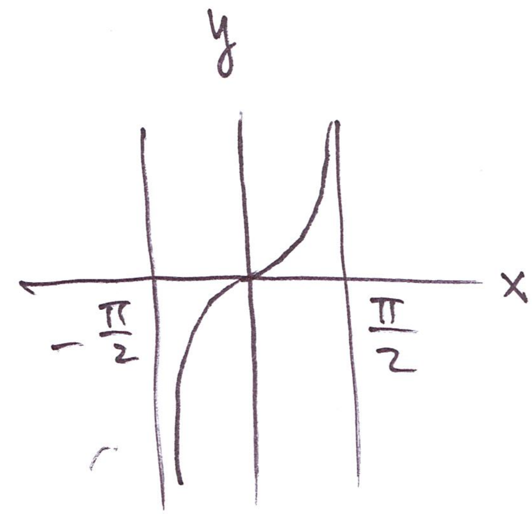
Ex. 8 Is $f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$ continuous at $(0,0)$?

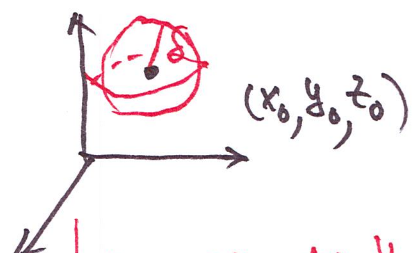
$$\lim_{(x,y)\to(0,0)} f = 0 = f(0,0)$$

Ex. 9 Where is $h(x,y) = \arctan\left(\dfrac{y}{x}\right)$ continuous?



$\underline{x \neq 0}$

- **Functions of three or more variables**



$0 < |(x,y,z) - (x_0,y_0,z_0)| < \delta$

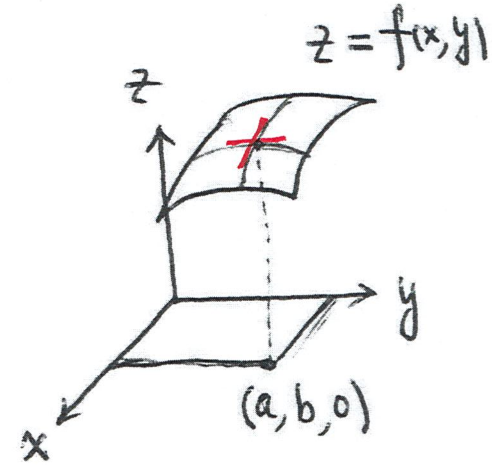
limit $\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$

continuity $f(x,y,z)$ is cont. at $(x_0,y_0,z_0) \Longleftrightarrow \displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} f = f(x_0,y_0,z_0)$

§14.3 Partial Derivatives

$$z = f(x, y) \qquad \frac{\partial f}{\partial x}(a,b) = \frac{d}{dx} f(x,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = \frac{d}{dy} f(a,y)$$



Examples

(1) $f(x,y) = x^2 + 3xy + y - 1$, $\frac{\partial f}{\partial x}(4,-5) = ?$ $\frac{\partial f}{\partial y}(4,-5) = ?$

$$\frac{\partial f}{\partial x}(4,-5) = \frac{d}{dx} f(x,-5)\Big|_{x=4} = \cancel{\frac{d}{dx}\left(2x+3y\right)\Big|_{x=4}} = \frac{d}{dx}\left(x^2 - 15x - 6\right)\Big|_{x=4} = (2x-15)\Big|_{x=4} = 8 - 15 = -7$$

$$\frac{\partial f}{\partial x}\Big|_{(4,-5)} = \left(2x + 3y\right)_{(4,-5)} = 2\cdot 4 + 3\cdot(-5) = 8 - 15 = -7$$

(2) $f(x,y) = y \sin(xy)$, $\frac{\partial f}{\partial x} = ?$ $\frac{\partial f}{\partial y} = ?$

$$\frac{\partial f}{\partial x} = \cancel{\frac{d}{dx}}\; y \cos(xy)\cdot y = y^2 \cos(xy), \qquad \frac{\partial f}{\partial y} = \sin(xy) + y\cdot\cos(xy)\cdot x$$

$$= \sin(xy) + xy\cos(xy)$$

(3) $f(x,y) = \dfrac{2y}{y+\cos x}$, $\underline{f_x} = ?$, $f_y = ?$ $\qquad \dfrac{\partial f}{\partial x} = f_x$

$$f_x = \left(2y\,(y+\cos x)^{-1}\right)_x = 2y\,(-1)\,(y+\cos x)^{-2}\cdot(-\sin x) = \frac{2y\sin x}{(y+\cos x)^2}$$

$$f_y = \frac{2\cdot(y+\cos x) - 2y\cdot 1}{(y+\cos x)^2} = \frac{2\cos x}{(\quad)^2}$$

$\underline{x}, \underline{y}$ – indep.

z – dep.

(4) $\underline{y\,z_{(x,y)} - \ln z^{(x,y)} = x + y}$, $\dfrac{\partial z}{\partial x} = ?$, $\dfrac{\partial z}{\partial y} = ?$

$$\frac{\partial}{\partial x}(yz - \ln z) = y\frac{\partial z}{\partial x} - \frac{1}{z}\cdot\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x+y) = 1$$

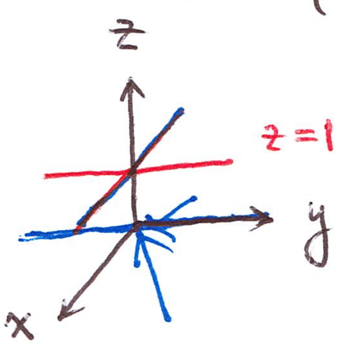
$$\frac{\partial}{\partial y}(yz - \ln z) = z + y\frac{\partial z}{\partial y} - \frac{1}{z}\cdot\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x+y) = 1$$

$$\begin{cases} \underline{y\dfrac{\partial z}{\partial x} - \dfrac{1}{z}\dfrac{\partial z}{\partial x} = 1} \\ z + y\dfrac{\partial z}{\partial y} - \dfrac{1}{z}\dfrac{\partial z}{\partial y} = 1 \end{cases}$$

$$\frac{\partial z}{\partial x} = \frac{1}{y - \frac{1}{z}}, \quad \frac{\partial z}{\partial y} = \frac{1-z}{y-\frac{1}{z}}$$

- <u>partial derivatives and continuity</u>

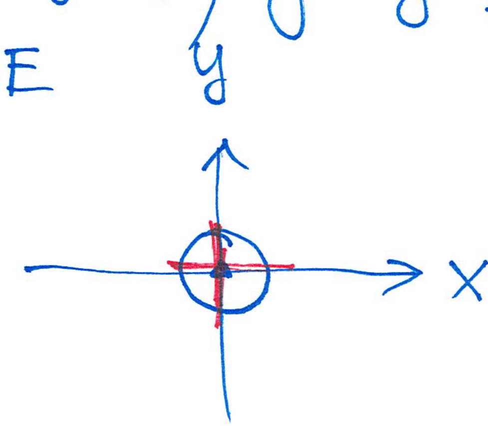
$$f(x,y) = \begin{cases} 0 & xy \neq 0 \\ 1 & \underline{xy = 0} \end{cases}$$



- $\underline{\lim\limits_{(x,y)\to(0,0)}} f(x,y) = \text{DNE}$

- $\underline{\dfrac{\partial f}{\partial x}(0,0)} = 0$

- $\dfrac{\partial f}{\partial y}(0,0) = 0$



$y = f(x)$

$f(x)$ is diff. at $x=a$ $\Longleftrightarrow$ $f'(a)$ exists

<u>Thrm</u> f is ~~cont.~~ diff. at $x=a$

$\Rightarrow$ f is cont at $x=a$

- higher-order derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \overset{?}{=} \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right), \quad \cdots$$

examples

(1) $f(x,y,z) = 1 - 2xy^2z + x^2y$, $\dfrac{\partial^4 f}{\partial y \partial x \partial y \partial z} = \dfrac{\partial^3}{\partial x \partial y \partial z}\left(-4xyz + x^2\right)$

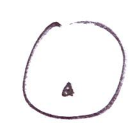
$$= \frac{\partial^2}{\partial y \partial z}\left(-4yz + 2x\right) = \frac{\partial}{\partial z}\left(-4z\right) = -4$$

(2) $w = \sqrt{u^2+v^2}$, $\dfrac{\partial^2 w}{\partial u^2} = ?$ $\dfrac{\partial^2 w}{\partial u \partial v} = ?$ $\dfrac{\partial^2 w}{\partial v^2} = \dfrac{u^2}{(\quad)^{\frac{3}{2}}}$

$$\frac{\partial w}{\partial u} = \frac{1}{2}(u^2+v^2)^{-\frac{1}{2}} \cdot 2u = u(u^2+v^2)^{-\frac{1}{2}} \qquad \frac{\partial^2 w}{\partial u \partial v} = u\left(-\frac{1}{2}\right)(u^2+v^2)^{-\frac{3}{2}} \cdot 2v = \frac{-uv}{(u^2+v^2)^{3/2}}$$

$$\frac{\partial^2 w}{\partial u^2} = (u^2+v^2)^{-\frac{1}{2}} + u\left(-\frac{1}{2}\right)(u^2+v^2)^{-\frac{3}{2}} \cdot 2u = (u^2+v^2)^{-\frac{1}{2}} - u^2(u^2+v^2)^{-\frac{3}{2}} = \frac{v^2}{(u^2+v^2)^{\frac{3}{2}}}$$

Clairaut's Thrm $f(x,y)$ is defined on a disk D and $(a,b) \in D$



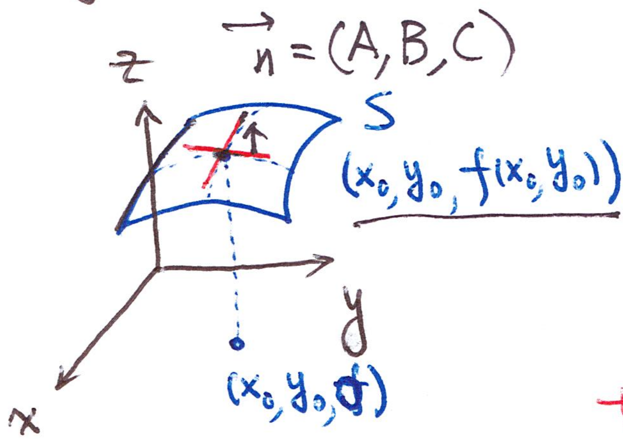
f_{xy} and f_{yx} are cont. on D $\Longrightarrow$ $f_{xy}(a,b) = f_{yx}(a,b)$

a,b)

§14.4 Tangent Planes and Linear Approximations

• tangent planes

Surface S : $z = f(x,y)$ has cont. partial derivatives



the equation of tangent plane to S at $(x_0, y_0, \overset{z_0}{\overset{\|}{f(x_0,y_0)}})$

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

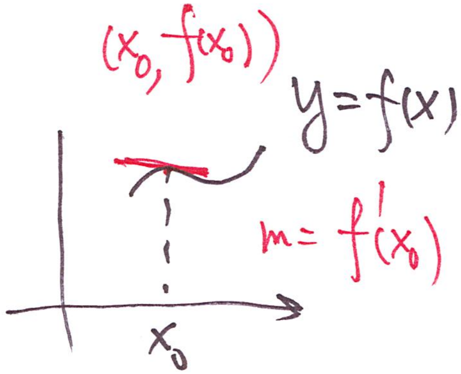
the eq of any plane passing through the point (x_0, y_0, z_0)

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$\Rightarrow \quad z - z_0 = -\frac{A}{C}(x - x_0) - \frac{B}{C}(y - y_0)$$

when $y = y_0$ $\quad z - z_0 = -\frac{A}{C}(x - x_0) \Longrightarrow -\frac{A}{C} = f_x(x_0,y_0)$

line eq with slope $f_x(x_0,y_0)$



$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$W(x,y,z) = z - f(x,y) = 0$$

$$\nabla W(x_0, y_0, z_0)$$