- implicit differentiation

$$F(x, y^{(x)}) = 0$$

$$0 = \frac{d}{dx} F(x, y)$$

$$= \frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot y'$$

$$\frac{dy}{dx} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

example (#28)

$$\cos(xy) = 1 + \sin y$$

$$\frac{dy}{dx} = ?$$

$$F(x,y,z^{(x,y)})=0$$

$\underbrace{x, y}_{\text{indep. var.}}$, z: dep. var.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z}$$

$$0=\frac{\partial}{\partial x}\left(F(x,y,z(x,y))\right)=\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} \Longrightarrow \frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$0=\frac{\partial}{\partial y}F(x,y,z(x,y))$$

example (#31) $x^2+2y^2+3z^2=1$, $\frac{\partial z}{\partial x}=?$, $\frac{\partial z}{\partial y}=?$

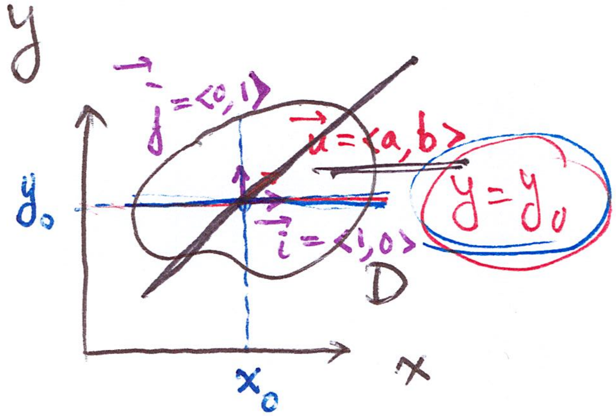
$$F=x^2+2y^2+3z^2 \cancel{-1}$$

$$\frac{\partial F}{\partial x}=2x,\quad \frac{\partial F}{\partial y}=4y,\quad \frac{\partial F}{\partial z}=6z$$

$$\frac{\partial z}{\partial x}=-\frac{2x}{6z}=-\frac{x}{3z},\quad \frac{\partial z}{\partial y}=-\frac{4y}{6z}=-\frac{2y}{3z}.$$

§14.6 Directional Derivative and the Gradient Vector

$$z = f(x, y), \quad (x, y) \in D \subset \mathbb{R}^2$$



$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{d}{dx} f(x, y_0)$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \frac{d}{dy} f(x_0, y)$$

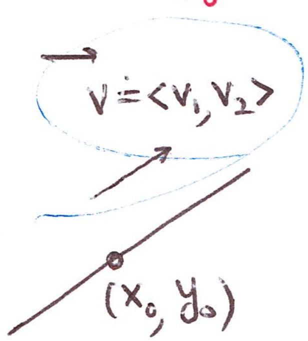
$$\vec{r}(t) = \langle x_0, y_0\rangle + t\langle 1, 0\rangle$$

$$= \langle x_0 + t, y_0\rangle$$

$$f(\vec{r}(t)) = f(x_0 + t, y_0)$$

$$\left.\frac{d}{dt} f(\vec{r}(t))\right|_{t=0} = \frac{\partial f}{\partial x}(x_0, y_0)$$

$x = x_0$



$$\vec{r}(t) = \langle x_0, y_0\rangle + t\langle v_1, v_2\rangle = \langle x_0 + tv_1, y_0 + tv_2\rangle$$

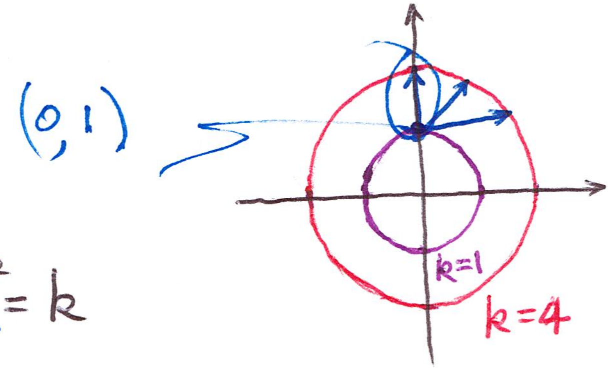
$$f(\vec{r}(t)) = f(x_0 + tv_1, y_0 + tv_2)$$

$$\left.\frac{df}{dt}\right|_{t=0} = \left[\frac{\partial f}{\partial x} \cdot \frac{d}{dt}(x_0 + tv_1) + \frac{\partial f}{\partial y} \cdot \frac{d}{dt}(y_0 + tv_2)\right]_{t=0}$$

$$= \frac{\partial f}{\partial x}(x_0, y_0)\, v_1 + \frac{\partial f}{\partial y}(x_0, y_0)\, v_2$$

<u>example</u> $z = f(x,y)$

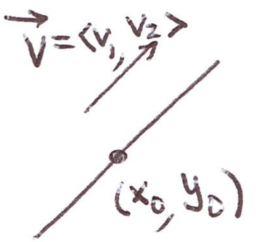
<u>level curves $z = k$</u> $\quad x^2 + y^2 = k$



$\nabla f(0,1) = \langle 0, 2 \rangle$

$$\nabla f = \langle 2x, 2y \rangle$$

- the line equation



$$\vec{r}(t) = \langle x_0, y_0 \rangle + t\vec{v}$$

$$= \langle x_0 + tv_1, y_0 + tv_2 \rangle$$

- the values of f on the line

$$f(\vec{r}(t)) = f(x_0 + tv_1, y_0 + tv_2)$$

$$\max_{\substack{\vec{v} \in \mathbb{R}^2 \\ |\vec{v}|=1}} \frac{d}{dt} f(\vec{r}(t)) \Big|_{t=0}$$

$$= \max_{|\vec{v}|=1} \left\{ \frac{\partial f}{\partial x}(\vec{r}(0)) \frac{dr_1(t)}{dt} + \frac{\partial f}{\partial y}(\vec{r}(0)) \frac{dr_2(t)}{dt} \right\}$$

$$= \max_{|\vec{v}|=1} \left\{ \frac{\partial f}{\partial x}(x_0, y_0)\, v_1 + \frac{\partial f}{\partial y}(x_0, y_0)\, v_2 \right\}$$

$$= \max_{|\vec{v}|=1} \left\{ \left\langle \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right\rangle \cdot \langle v_1, v_2 \rangle \right\}$$

$$= \max_{|\vec{v}|=1} \left| \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle (x_0, y_0) \right| |\vec{v}| \cos\theta$$

$$= \left| \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \Big|_{(x_0, y_0)} \right| \quad \text{if } \theta = 0^\circ$$

- directional derivatives along the direction $\vec{u} = \langle a, b \rangle$ $\quad |\vec{u}| = 1$

$$\frac{d}{dt} f(\vec{r}_0 + t\vec{u})\Big|_{t=0} = D_{\vec{u}} f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\, a + \frac{\partial f}{\partial y}(x_0, y_0)\, b = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle\Big|_{(x_0,y_0)} \cdot \langle a, b \rangle$$

$$= \nabla f(x_0, y_0) \cdot \cancel{\langle a, b \rangle}\, \vec{u}$$

$\vec{u} = \vec{i} = \langle 1, 0 \rangle$

$\vec{u} = \vec{j} = \langle 0, 1 \rangle$

- gradient vector

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

- functions of three variables $\quad \vec{u} = \langle a, b, c \rangle$

$w = f(x, y, z)$

$$D_{\vec{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \vec{u} \qquad \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

examples Find the gradient of f, directional derivative along $\vec{u}$

(1) $f(x,y) = xe^{y} + \cos(xy)$, $P(2,0)$, $\vec{u} = \langle 3, -4 \rangle$

$$\nabla f\Big|_{(2,0)} = \langle e^{y} - \sin(xy)\cdot y,\ xe^{y} - x\sin(xy) \rangle\Big|_{(2,0)} = \langle 1, 2 \rangle$$

$$D_{\vec{u}} f(2,0) = \nabla f(2,0) \cdot \frac{\vec{u}}{|\vec{u}|} = \langle 1, 2 \rangle \cdot \frac{\langle 3, -4 \rangle}{\sqrt{9+16}} = \frac{3 + (-8)}{5} = -1$$

(2)(#9) $f(x,y,z) = x^2yz - xyz^3$, $P(2,-1,1)$, $\vec{u} = \langle 0, \frac{4}{5}, -\frac{3}{5} \rangle$

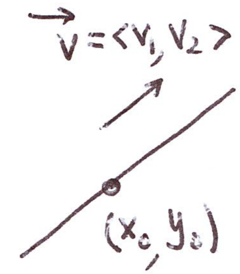
$$\frac{\partial f}{\partial x} = 2xyz - yz^3,\quad \frac{\partial f}{\partial y} = x^2z - xz^3,\quad \frac{\partial f}{\partial z} = x^2y - 3xyz^2,\quad \nabla f \cancel{(2,-1,1)} = \langle$$

$$\nabla f(2,-1,1) = \langle -4+1,\ 4-2,\ -4+6 \rangle = \langle -3, 2, 2 \rangle$$

$$D_{\vec{u}} f(2,-1,1) = \langle -3, 2, 2 \rangle \cdot \langle 0, \frac{4}{5}, -\frac{3}{5} \rangle = \frac{8}{5} - \frac{6}{5} = \frac{2}{5}$$

(3)(#16) $f(x,y,z) = \sqrt{xyz}$, $P(3,2,6)$, $\vec{u} = \langle -1, -2, 2 \rangle$

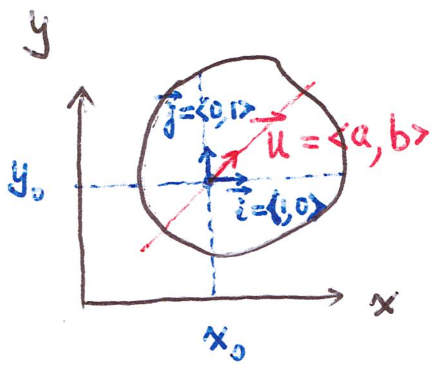
§14.6 Directional Derivative and the Gradient Vector



the equation of line

$\vec{r}(t) =$

$$z = f(x,y), \quad (x,y) \in D \subset R^2$$



$$\frac{\partial f}{\partial x}(x_0, y_0)$$

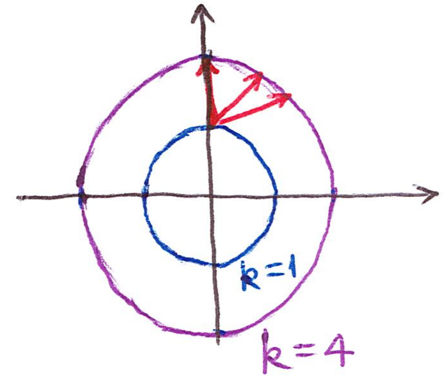
$$\frac{\partial f}{\partial y}(x_0, y_0)$$

Question In which direction at (x_0, y_0) does the value of $f(x,y)$ increase most rapidly?

$\Longleftrightarrow$ Find $\vec{u} = \langle a,b \rangle$ such that

$$\frac{d}{dt} f\left((x_0,y_0) + t\vec{u}\right)\Big|_{t=0} = \max_{\substack{\vec{v} \in R \\ |\vec{v}|=1}} \frac{d}{dt} f\left((x_0,y_0) + t\vec{v}\right)\Big|_{t=0}$$

example $z = f(x,y)$



level curves $(z=k)$ $\quad x^2+y^2=k$

$$\max_{|\vec{v}|=1} \frac{d}{dt} f\left(\vec{r}(t)\right)\Big|_{t=0} \qquad \vec{r}(t) = \langle x_0, y_0 \rangle + t\vec{v}$$

$=$

Thrm Assume that f is differentiable.

$$\Longrightarrow \max_{|\vec{v}|=1} \left| D_{\vec{v}} f(x,y) \right| = \left| D_{\vec{u}} f \right| = \nabla f, \quad \text{where } \vec{u} = \nabla f.$$