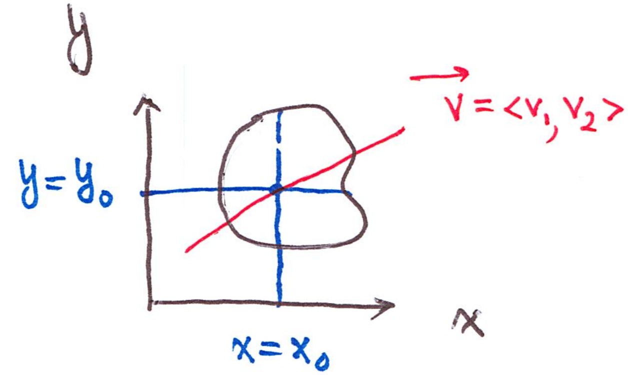
§14.6 Directional Derivative and the Gradient Vector

$z = f(x, y)$, $(x, y) \in D$



partial derivative

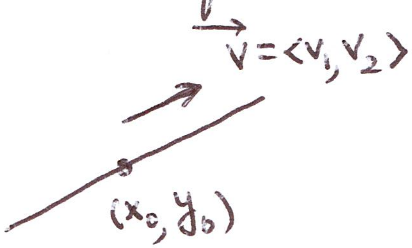
$$\frac{\partial f}{\partial x}(x_0, y_0) =$$

$$\frac{\partial f}{\partial y}(x_0, y_0) =$$

directional derivative along $\vec{v}$

$$\left. \frac{d}{dt} f(\vec{r}(t)) \right|_{t=0}$$

the eq of the line



$\vec{r}(t) =$

the restriction of f on the line

$f(\vec{r}(t)) =$

$=$

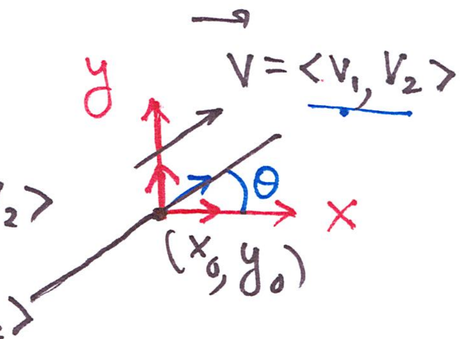
- directional derivative along the direction $\vec{v} = \langle v_1, v_2 \rangle$

$$D_{\vec{v}} f(x_0, y_0) = \frac{d}{dt} f(\vec{r}(t)) \Big|_{t=0} = \frac{d}{dt} f(x_0 + tv_1, y_0 + tv_2) \Big|_{t=0} = \frac{\partial f}{\partial x}(x_0, y_0)\, v_1 + \frac{\partial f}{\partial y}(x_0, y_0)\, v_2$$

$$= \nabla f(x_0, y_0) \cdot \vec{v}$$

$\vec{v} = \vec{i} = \langle 1, 0 \rangle$ or $\vec{v} = \vec{j} = \langle 0, 1 \rangle$

$$\vec{r}(t) = \langle x_0, y_0 \rangle + t \langle v_1, v_2 \rangle$$
$$= \langle x_0 + tv_1, y_0 + tv_2 \rangle$$



- gradient vector

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

- functions of three variables

$w = f(x, y, z)$, $\nabla f = \langle \qquad \rangle$, $D_{\vec{v}} f(x_0, y_0, z_0) =$

<u>examples</u> Find the gradient of, directional derivative along $\vec{u}$

(1) $f(x,y) = xe^{y} + \cos(xy)$, $P(2,0)$, $\vec{u} = \langle 3, -4 \rangle$

(2) (#9) $f(x,y,z) = x^2yz - xyz^3$, $P(2,-1,1)$, $\vec{u} = \langle 0, \frac{4}{5}, -\frac{3}{5} \rangle$

Question In which direction at (x_0, y_0) does the value of $f(x,y)$ increase most rapidly?

$\Longleftrightarrow$ Find $\vec{u} = \langle a, b \rangle$ such that

$$\frac{d}{dt} f\left(\langle x_0, y_0 \rangle + t\vec{u}\right)\Big|_{t=0} = \max_{\substack{\vec{v} \in \mathbb{R}^2 \\ |\vec{v}|=1}} \frac{d}{dt} f\left(\langle x_0, y_0\rangle + t\vec{v}\right)\Big|_{t=0}$$

$$= \max_{|\vec{v}|=1} \left\{ \nabla f(x_0, y_0) \cdot \vec{v} \right\}$$

$$= \max_{|\vec{v}|=1} |\nabla f(x_0, y_0)|\, \cancel{|\vec{v}|} \cos\theta$$

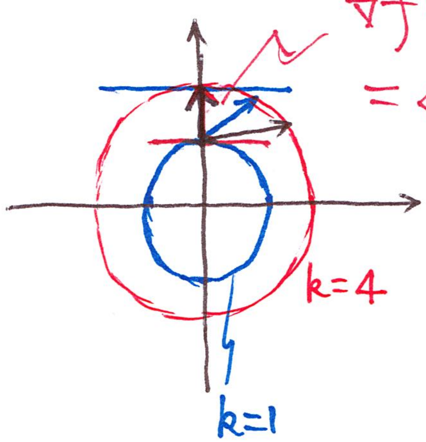
$$= |\nabla f(x_0, y_0)| \max_{|\vec{v}|=1} \cos\theta$$

$$= |\nabla f(x_0, y_0)| \quad \text{when } \theta = 0$$

$\Longleftrightarrow \vec{v} \,/\!/\, \nabla f$

example $z = f(x,y) = x^2 + y^2$

level curve $x^2 + y^2 = k$



$\nabla f(0,1) = \langle 2x, 2y \rangle\big|_{(0,1)} = \langle 0, 2 \rangle$

Theorem Assume that f is differentiable

$$\Longrightarrow \max_{|\vec{v}|=1} D_{\vec{v}} f(x,y) = |\nabla f|$$

it occurs when $\vec{v}$ is in the same direction as ∇f.

examples find the maximum rate of change of f at the given point and the direction in which it occurs.

(3) (#22) $f(s,t) = te^{st}$, $(0,2)$

$$\nabla f = \langle te^{st}\cdot t,\ e^{st} + te^{st}\cdot s\rangle = \langle t^2 e^{st},\ (1+st)e^{st}\rangle$$

$$\nabla f(0,2) = \langle 4, 1\rangle$$

$$|\nabla f(0,2)| = |\langle 4,1\rangle| = \sqrt{16+1} = \sqrt{17}$$

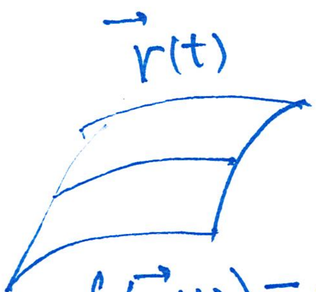
(4) (#24) $f(x,y,z) = \frac{x+y}{z}$, $(1,1,-1)$, $\nabla f = \langle \frac{1}{z}, \frac{1}{z}, -(x+y)z^{-2}\rangle$

$$\nabla f(1,1,-1) = \langle -1, -1, -2\rangle = -\langle 1,1,2\rangle$$

$$|\nabla f(1,1,-1)| = |\langle 1,1,2\rangle| = \sqrt{1+1+4} = \sqrt{6}$$

- gradients and tangents to level sets (curves or surfaces)

level surface $\quad S = \{ (x,y,z) \mid \underline{f(x,y,z)=k} \}$



$f(\vec{r}(t)) = f(x(t), y(t), z(t))$

$\boxed{\nabla f \perp S}$

$\forall$ smooth curve C on S : $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\Rightarrow \boxed{f(\vec{r}(t)) = k}$

$\Rightarrow 0 = \dfrac{d}{dt} f(\vec{r}(t)) = \dfrac{\partial f}{\partial x} \cdot x'(t) + \dfrac{\partial f}{\partial y} \cdot y'(t) + \dfrac{\partial f}{\partial z} \cdot z'(t)$

$= \nabla f \cdot \vec{r}\,'(t)$

$\Rightarrow \nabla f \perp \vec{r}\,'(t)$

- equations of tangent planes at $(x_0, y_0, z_0) \in S$

(1) level surface $\quad f(x,y,z) = k$

$\vec{n} = \nabla f(x_0, y_0, z_0) \Longrightarrow 0 = \dfrac{\partial f}{\partial x}(x_0,y_0,z_0)(x-x_0) + \dfrac{\partial f}{\partial y}(x_0,y_0,z_0)(y-y_0) + \dfrac{\partial f}{\partial z}(\vec{x}_0)(z-z_0)$

$= \langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z} \rangle (x_0, y_0, z_0)$

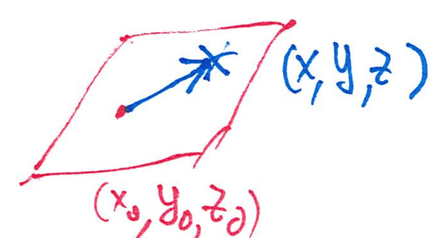
$= \nabla f(x_0,y_0,z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle$

(2) graph $\quad z = f(x,y)$

$z = f(x_0,y_0) + \dfrac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \dfrac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$

$\boxed{0 = g(x,y,z)} = z - f(x,y)$

$\nabla g = \langle -\dfrac{\partial f}{\partial x}, -\dfrac{\partial f}{\partial y}, 1 \rangle$



- equations of normal lines at $(x_0, y_0, z_0) \in S$

$\nabla f(x_0, y_0, z_0)$

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\, \nabla f(x_0, y_0, z_0)$$

examples find the equations of the tangent plane and normal line

(1) $f(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3, \quad P(-2, 1, -3)$

$$\nabla f(-2,1,-3) = \left\langle \frac{1}{2}x, 2y, \frac{2}{9}z \right\rangle \Big|_{(-2,1,-3)} = \left\langle -1, 2, -\frac{2}{3} \right\rangle$$

$$0 = \left\langle -1, 2, -\frac{2}{3} \right\rangle \cdot \langle x+2, y-1, z+3 \rangle = -(x+2) + 2(y-1) - \frac{2}{3}(z+3)$$

$$= -x + 2y - \frac{2}{3}z - 6 = 0$$

$$\vec{r}(t) = \langle -2, 1, -3 \rangle + t \left\langle -1, 2, -\frac{2}{3} \right\rangle$$

(2) $f = xyz^2 = 6, \quad P(3, 2, 1)$

$$\nabla f = \langle yz^2, xz^2, 2xyz \rangle, \quad \nabla f(3,2,1) = \langle 2, 3, 12 \rangle$$

$$0 = \langle 2, 3, 12 \rangle \cdot \langle x-3, y-2, z-1 \rangle$$

$$\vec{r}(t) = \langle 3, 2, 1 \rangle + t \langle 2, 3, 12 \rangle$$