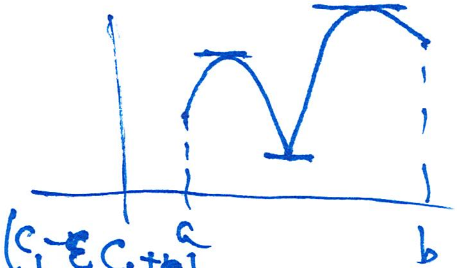
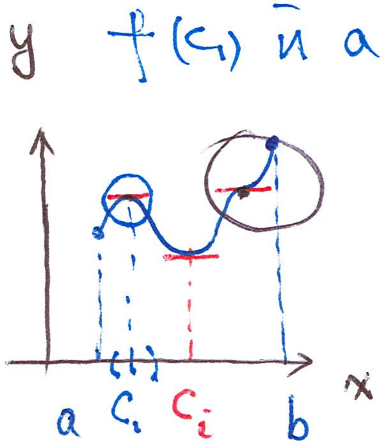
§14.7 Maximum and Minimum Values

$f(c_1)$ is a local max. $\iff f(c_1) \geq f(x)$

$\forall\, x \in D_\varepsilon(c_1) = (c_1-\varepsilon, c_1+\varepsilon)$





$$\max_{x\in[a,b]}/\min_{x\in[a,b]} f(x) = \max/\min \left\{ f(a),\ f(b),\ \underline{f(c_i)} \right\}$$

$\{a, b\}$ — boundary points

$\{c_i\}$ — critical points (interior)

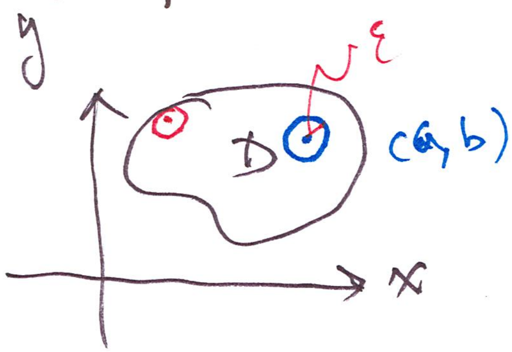
$f'(c_i) = 0$, c_i —

$f'(c_i)$ DNE

Def. (extreme values)

(1) $f(a,b)$ is a local maximum value of $f \iff f(a,b) \geq f(x,y) \quad \forall\, (x,y) \in D_\varepsilon(a,b)$

(2) $f(a,b)$ is a local minimum value of $f \iff f(a,b) \leq f(x,y) \quad \forall\, (x,y) \in D_\varepsilon(a,b)$



$$D_\varepsilon(a,b) = \left\{ (x,y) \;\middle|\; \underbrace{\left| (x,y) - (a,b) \right|}_{= \sqrt{(x-a)^2 + (y-b)^2}} < \varepsilon \right\}$$

Thrm (1st derivative test)

f has a local maximum/minmum at an interior point (a,b) and $f_x(a,b)$ and $f_y(a,b)$ exist $\Longrightarrow \nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle = \langle 0,0\rangle$

$(a, b, f(a,b))$

Proof Assume that $f(a,b)$ is a local maximum value of f

$\overset{?}{\Longrightarrow} \dfrac{\partial f}{\partial x}(a,b) = 0, \quad \dfrac{\partial f}{\partial y}(a,b) = 0$

$\begin{cases} z = f(x,y) \\ y = b \end{cases} \Rightarrow z = f(x,b) = g(x)$

$g(a)$ is a local max. of $g(x) = f(x,b)$

$\Rightarrow 0 = g'(a) = \dfrac{\partial f}{\partial x}(a,b)$

Def.

- (a,b) is a critical point of $f \iff \nabla f(a,b) = \langle 0,0\rangle$ or $\nabla f(a,b)$ DNE

- (a,b) is a saddle point of $f \iff \nabla f(a,b) = \langle 0,0\rangle$ or $\nabla f(a,b)$ DNE but $f(a,b)$ is not a local max./min.

examples Find extreme values

(1) $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

$= (x-1)^2 + (y-3)^2 + 4$

$\nabla f(x,y) = \langle 2x-2,\ 2y-6 \rangle = \langle 0,0 \rangle \Rightarrow \begin{cases} 2x-2=0 \\ 2y-6=0 \end{cases} \Rightarrow (1,3)$

$f(1,3) = 4 \le (x-1)^2 + (y-3)^2 + 4 = f(x,y)$

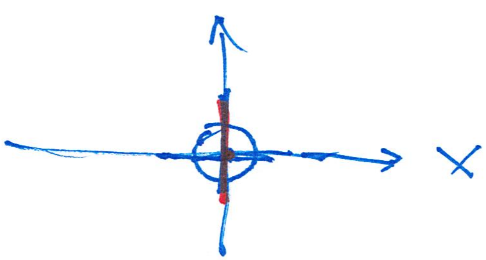
abs. min

(2) $f(x,y) = y^2 - x^2$

$\nabla f(x,y) = \langle -2x,\ 2y \rangle = \langle 0,0 \rangle \Rightarrow \begin{cases} -2x=0 \\ 2y=0 \end{cases} \Rightarrow (0,0)$

$f(0,0) = 0$

$f(0,0)=0$ is not a local min



$f(x,0) = -x^2 < 0$ for $x \neq 0$

$f(0,y) = y^2 > 0$ for $y \neq 0$

Thrm (2nd derivative test)

Assume that f_{xx}, f_{yy}, and f_{xy} are continuous on a disk with center (a,b)

and that $\nabla f(a,b) = \langle 0, 0 \rangle$.

Let $D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}_{(a,b)}$

$\Longrightarrow$ (a) $\boxed{D > 0}$ and $f_{xx}(a,b) > 0 \Longrightarrow f(a,b)$ is a local minimum ~~maximum~~.

$\Rightarrow f_{yy}(a,b) > 0$

(b) $\boxed{D > 0}$ and $f_{xx}(a,b) < 0 \Longrightarrow f(a,b)$ is a local maximum.

(c) $D < 0$. $\Longrightarrow f(a,b)$ is not a local max./min.

Ex. 3 Find the local max. and min. values and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$ $\vec{u}$

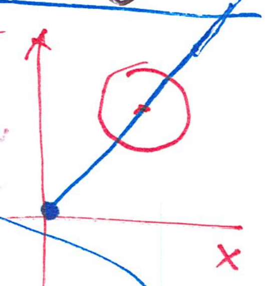
$\nabla f = \langle 4x^3 - 4y,\ 4y^3 - 4x \rangle = \langle 0, 0 \rangle$

$$\begin{cases} x^3 - y = 0 \Rightarrow y = x^3 \\ y^3 - x = 0 \end{cases}$$

$D_{\vec{u}} f(a,b)$

$= \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$

$D_{\vec{x}} f(a,b)$



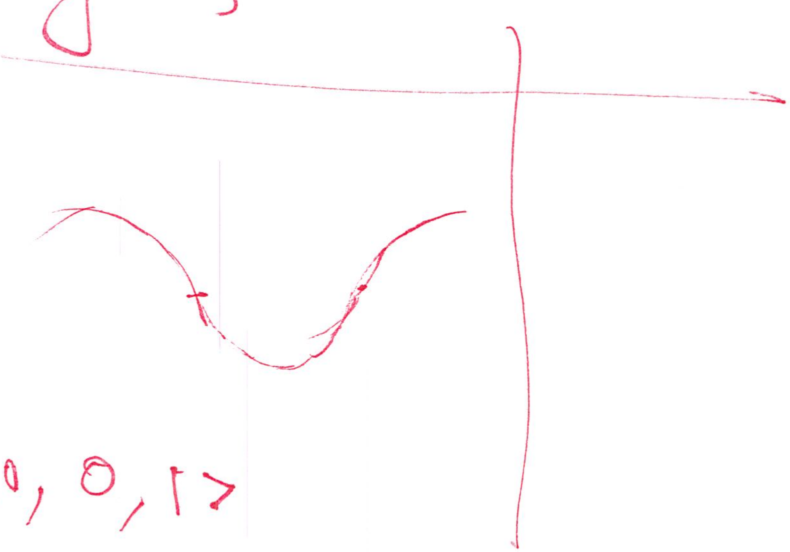
$$D = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}^2(a,b) > 0$$

$$\underbrace{f_{xx}(a,b)}_{>0}\, f_{yy}(a,b) > f_{xy}^2(a,b) \geq 0$$

$a \quad b$

$$ax = b \qquad \begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

$$cy = d$$



$$z - f(x,y) = 0$$

$$\left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle \Longrightarrow \langle 0, 0, 1 \rangle$$