Chapter 15 Multiple Integrals (7 lectures)

§15.1 Double Integrals over Rectangles

- **Review of the Definite Integral**

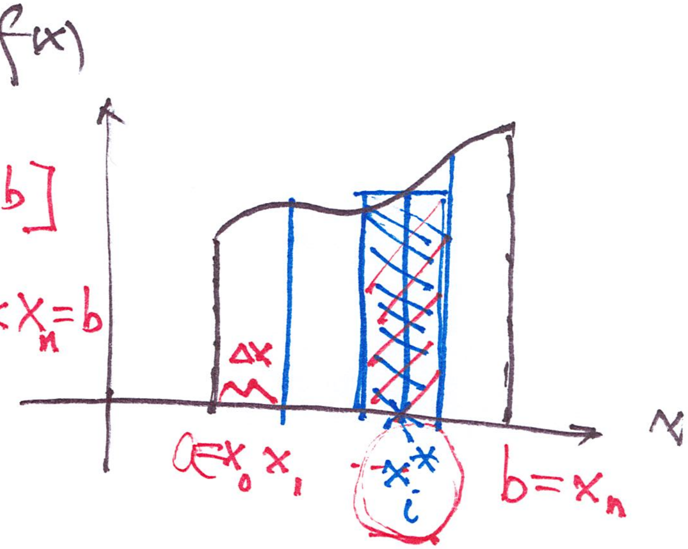
$$\int_a^b f(x)\,dx = \underbrace{\lim_{\substack{\Delta x \to 0 \\ \text{or } n\to\infty}} \sum_{i=1}^{n} f(x_i^*)\,\Delta x}_{\text{Riemann Sum}}$$

(1) partition of $[a,b]$

$a = x_0 < x_1 < x_2 < \dots < x_n = b$

$x_i = a + i\Delta x$

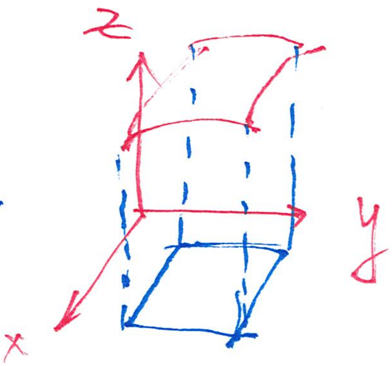
$\Delta x = \dfrac{b-a}{n}$



$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\,\underline{\Delta x}$$

- **Volume and Double Integral**

$f(x,y)$ on $\underline{R = [a,b]\times[c,d]}$

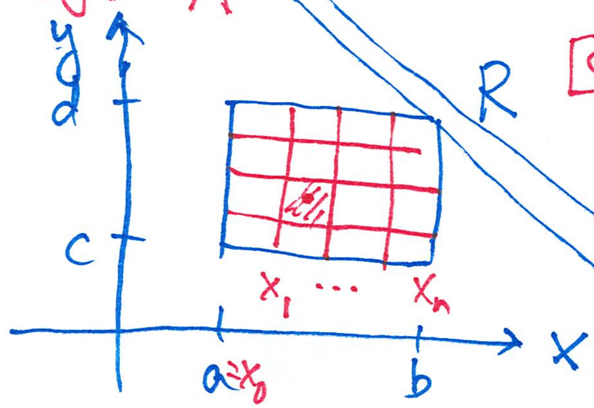


$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i,j} \underline{f(x_i^*, y_j^*)}\,\Delta x\,\Delta y$$

$[a,b]: \quad a = x_0 < x_1 < \dots < x_n = b$

$[c,d]: \quad c = y_0 < y_1 < \dots < y_m = d$

$[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ $\quad (x_i^*, y_j^*)$

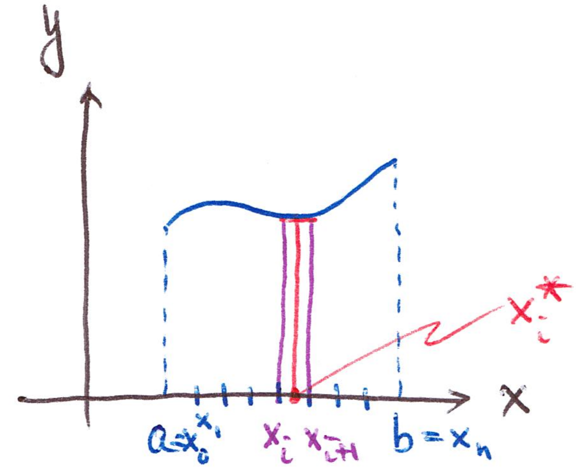


Chapter 15 Multiple Integrals (7 lectures)

§15.1 Double Integrals over Rectangles

- **Review of the Definite Integral**

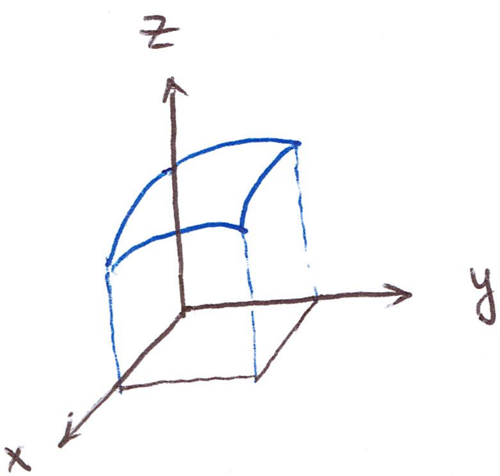
$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_i f(x_i^*)\,\Delta x$$



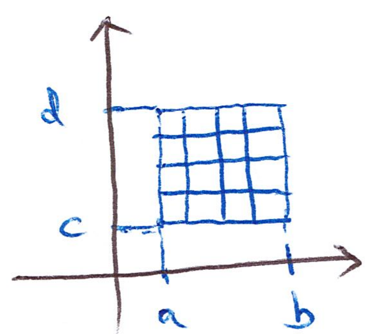
$$\Delta x = \frac{b-a}{n}$$

a partition of interval $[a,b]$: $x_i = x_0 + i\,\Delta x$

- **Volume and Double Integral**



a partition of rectangle $[a,b]\times[c,d] = R$



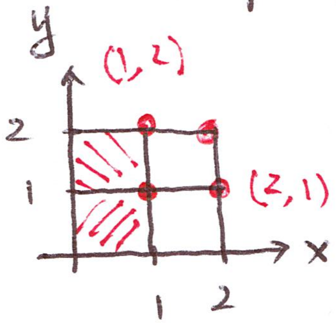
$$V = \iint_R f(x,y)\,dA =$$

$a = x_0 < x_1 < \cdots < x_m = b$

$c = y_0 < y_1 < \cdots < y_n = d$

Examples

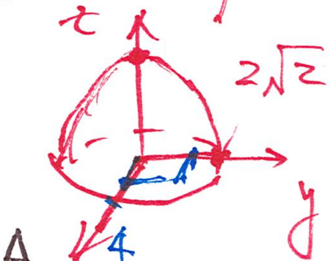
(1) Estimate the volume of the solid that lies above the square $R=[0,2]\times[0,2]$ and below the elliptic paraboloid $z=16-x^2-2y^2$. Divide R into 4 equal squares and choose the sample point to the upper right corner of each square R_{ij}. (see Fig. 8 too)



$$f(1,1)\cdot 1\cdot 1+f(2,1)\cdot 1\cdot 1+f(1,2)\cdot 1\cdot 1+f(2,2)\cdot 1\cdot 1$$

$$=(16-1-2)+(16-4-2)+(16-1-8)+(16-2^2-2\cdot 2^2)$$

$$=13+10+7+4=34$$



(2) $R=\{(x,y)\mid -1\le x\le 1,\ -2\le y\le 2\}$, evaluate the integral $\iint_R \sqrt{1-x^2}\,dA$

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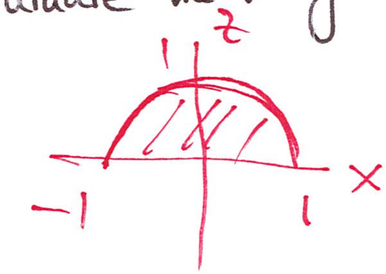
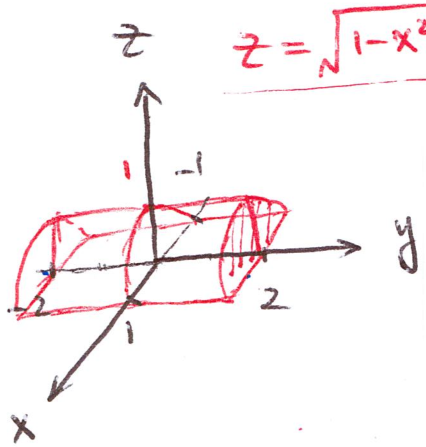
$$\frac{1}{2}\pi\cdot 1^2\cdot 4$$

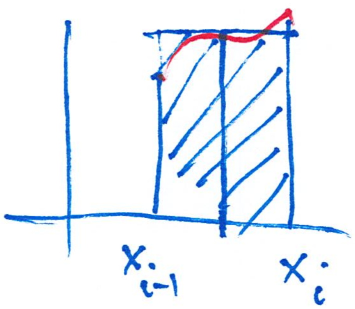
$$=2\pi$$

$z=\sqrt{1-x^2} \Longrightarrow z^2+x^2=1$





• the midpoint rule



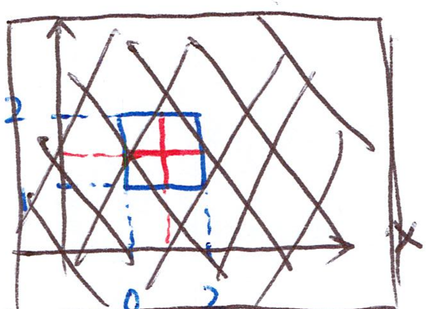
$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\,\Delta A,$$

$(\bar{x}_i, \bar{y}_j)$ is the center of $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$

$$\bar{x}_i = \frac{x_{i-1}+x_i}{2}, \quad \bar{y}_j = \frac{y_{j-1}+y_j}{2}$$

Ex. 3 Use the midpoint rule with $m=n=2$ to estimate the value of the integral

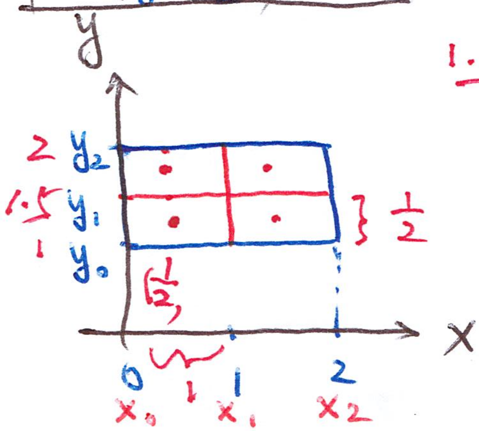
$$\iint_R (x-3y^2)\,dA, \text{ where } R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}.$$



$$\approx f\left(\frac{1}{2}, \frac{5}{4}\right)\cdot\frac{1}{2}\cdot 1 + f\left(\frac{3}{2}, \frac{5}{4}\right)\cdot\frac{1}{2}\cdot 1 + f\left(\frac{1}{2}, \frac{7}{4}\right)\cdot\frac{1}{2}\cdot 1$$

$$+ f\left(\frac{3}{2}, \frac{7}{4}\right)\cdot\frac{1}{2}\cdot 1$$

$$\frac{1.5+1}{2} = \frac{\frac{3}{2}+1}{2} = \frac{5}{4}$$



• ~~§15.2~~ Iterated Integral

Fubini's Thrm If f is continuous on the rectangle $R=\{(x,y) \mid a\le x\le b,\ c\le y\le d\}$.

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

Remark This is true if f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Examples

(1) $\iint_R (x-3y^2)\,dA$, $R=[0,2]\times[1,2]$

$$=\int_0^2\left[\int_1^2 (x-3y^2)\,dy\right]dx = \int_0^2 \Big[xy-y^3\Big]_{y=1}^{2} dx = \int_0^2 \Big[(2x-8)-(x-1)\Big]dx$$

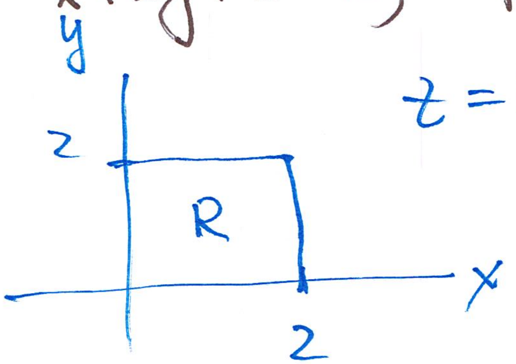
$$=\int_0^2 [x-7]\,dx = \frac{1}{2}x^2-7x\Big]_{x=0}^{2} = \frac{1}{2}\cdot 2^2-7\cdot 2 = 2-14=-12$$

(2) $\displaystyle\iint_{R=[1,2]\times[0,\pi]} y\sin(xy)\,dA = \int_0^{\pi}\left(\int_1^2 y\sin(xy)\,dx\right)dy$

$$= \int_0^{\pi}\Big[-\cos(xy)\Big]_{x=1}^{2}\,dy = \int_0^{\pi}\left[\cos y - \cos 2y\right]dy$$

$$= \left[\sin y - \frac{1}{2}\sin(2y)\right]_0^{\pi} = 0$$

(3) Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2+2y^2+z=16$, the planes $x=2$ and $y=2$, and the three coordinate planes.



$z=16-x^2-2y^2$ $\qquad \displaystyle\iint_R (16-x^2-2y^2)\,dA$

$$= \int_0^2 dx\int_0^2 (16-x^2-2y^2)\,dy = \int_0^2 \left[16y - x^2y - \frac{2}{3}y^3\right]_{y=0}^{2} dx$$

$$= \int_0^2\left[(16-x^2)\cdot 2 - \frac{2}{3}\cdot 8\right]dx = \left(32-\frac{16}{3}\right)x - \frac{2}{3}x^3\Big|_0^2 = 64 - \frac{32}{3} - \frac{2}{3}\cdot 8$$

$$\underline{f(x,y)} = \underline{g(x)\,h(y)} \implies \iint_R f(x,y)\,dA = \left[\int_a^b g(x)\,dx\right]\cdot\left[\int_c^d h(y)\,dy\right]$$

$$\parallel$$

$$\int_a^b dx \int_c^d \underline{g(x)}\,h(y)\,dy = \int_a^b g(x)\,\underline{\left[\int_c^d h(y)\,dy\right]}\,dx$$

<u>Ex. 5</u>

$$\iint_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} \sin x \cos y \, dA$$

$$= \int_0^{\frac{\pi}{2}} \sin x \, dx \int_0^{\frac{\pi}{2}} \cos y \, dy$$

$$= \Big[-\cos x\Big]_0^{\frac{\pi}{2}} \Big[\sin y\Big]_0^{\frac{\pi}{2}} =$$