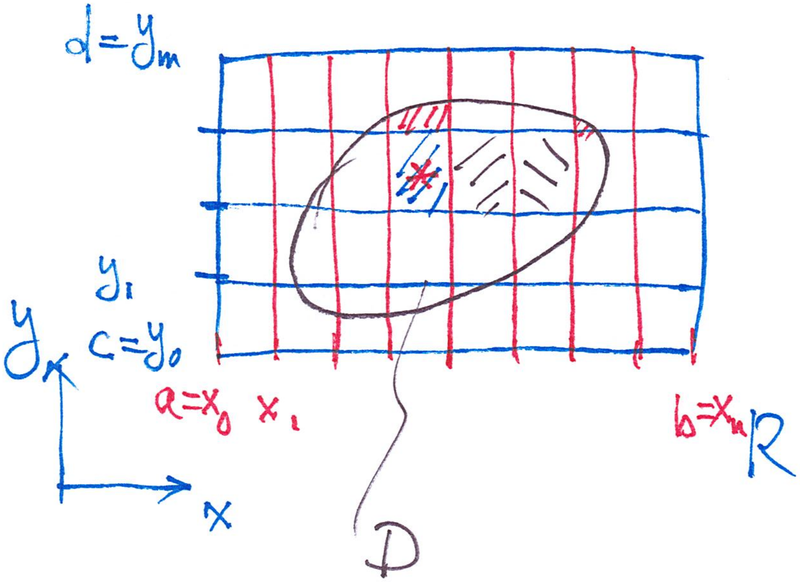
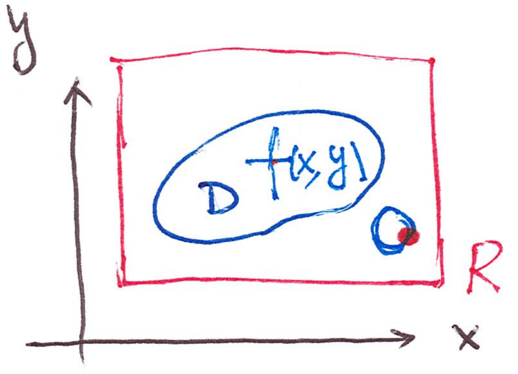
$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i,j} f(x_i^*, y_j^*)\,\Delta x\,\Delta y$$

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx \quad \begin{cases} \Delta x \to 0 \\ \Delta y \to 0 \end{cases}$$



$$[x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

§15.2 Double Integrals over General Regions



general region

$$\iint_D f(x,y)\,dA = \iint_R \hat{f}(x,y)\,dA = \int_a^b\left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right)dx$$

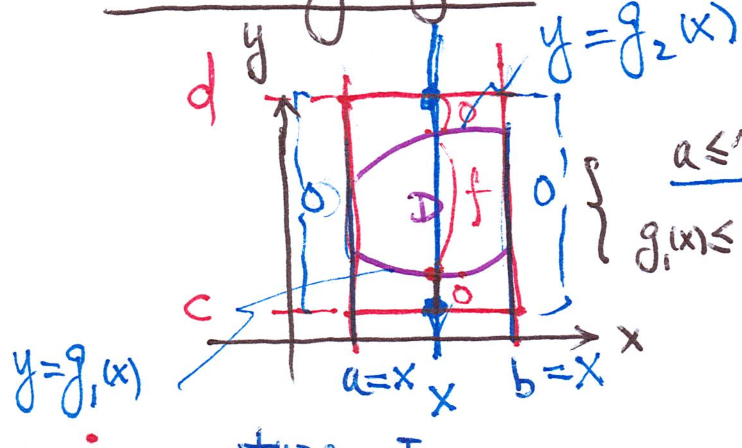
$$= \int_a^b\left(\int_c^d \hat{f}(x,y)\,dy\right)dx$$

$$= \int_a^b\left[\int_c^{g_1(x)} \cancel{0}\,dy + \int_{g_1(x)}^{g_2(x)} f\,dy + \int_{g_2(x)}^{d} \cancel{0}\,dy\right]dx$$

$$\hat{f}(x,y) = \begin{cases} f(x,y), & \forall (x,y)\in D \\ 0 & \forall (x,y)\in R\setminus D \end{cases}$$

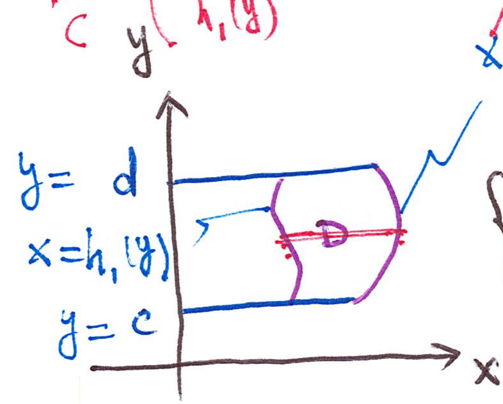
$$\iint_D f\,dA = \int_c^d\left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)dy$$

- elementary regions



$$\begin{cases} a\le x\le b \\ g_1(x)\le y\le g_2(x) \end{cases}$$

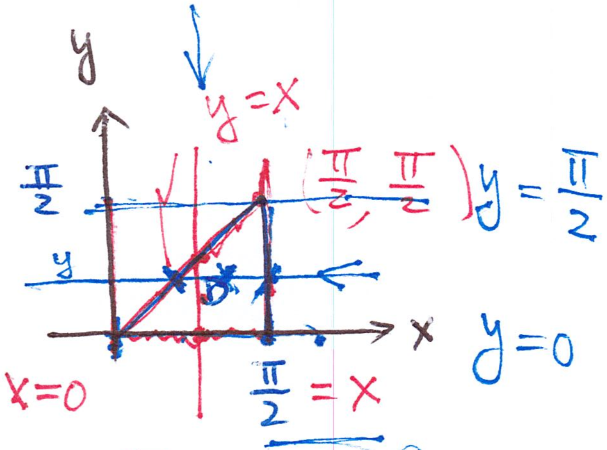
type-I



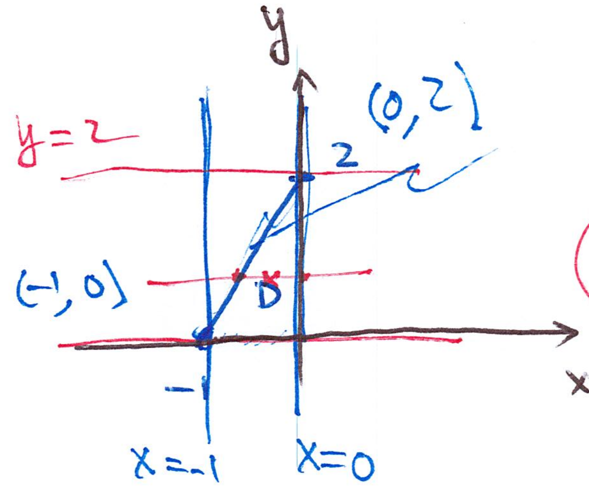
$$\begin{cases} c\le y\le d \\ h_1(y)\le x\le h_2(y) \end{cases}$$

type-II

__examples__ express the following regions as type-I and/or type II regions



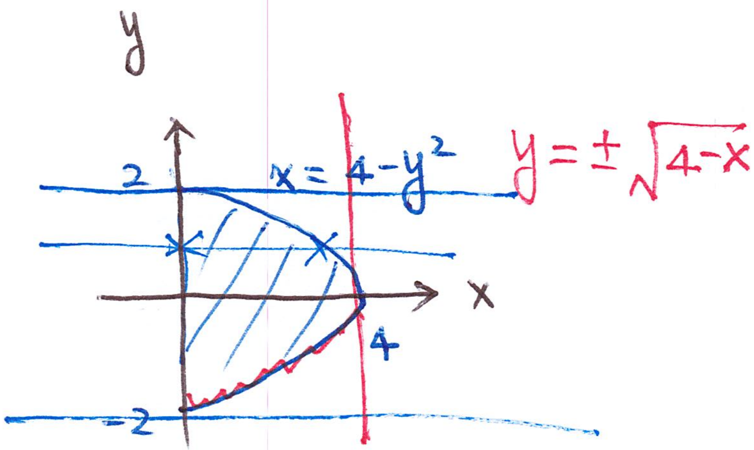
$$\begin{cases} 0 \le x \le \frac{\pi}{2} \\ 0 \le y \le x \end{cases} \qquad \begin{cases} y \le x \le \frac{\pi}{2} \\ 0 \le y \le \frac{\pi}{2} \end{cases}$$



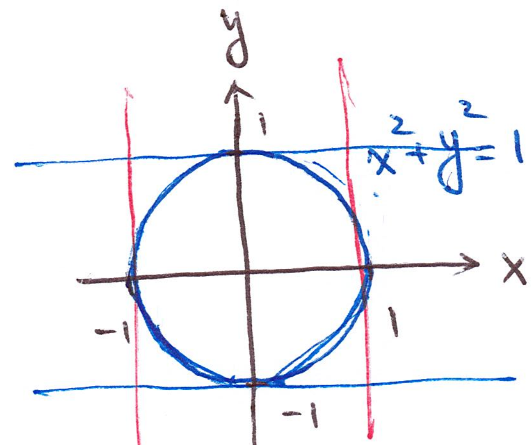
$$\frac{y-2}{0-2} = \frac{x-0}{-1-0}$$

$$y = 2 + 2x$$

$$\begin{cases} -1 \le x \le 0 \\ 0 \le y \le 2+2x \end{cases} \qquad \begin{cases} \frac{y-2}{2} \le x \le 0 \\ 0 \le y \le 2 \end{cases}$$



$$\begin{cases} 0 \le x \le 4 \\ -\sqrt{4-x} \le y \le \sqrt{4-x} \end{cases} \qquad \begin{cases} 0 \le x \le 4-y^2 \\ -2 \le y \le 2 \end{cases}$$



$$\begin{cases} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \end{cases} \qquad \begin{cases} -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \\ -1 \le y \le 1 \end{cases}$$

- D is type-I

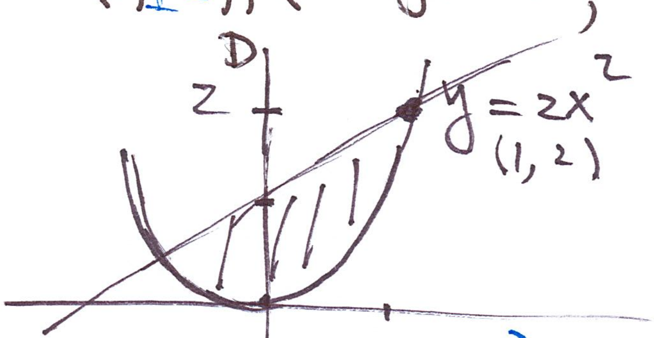
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

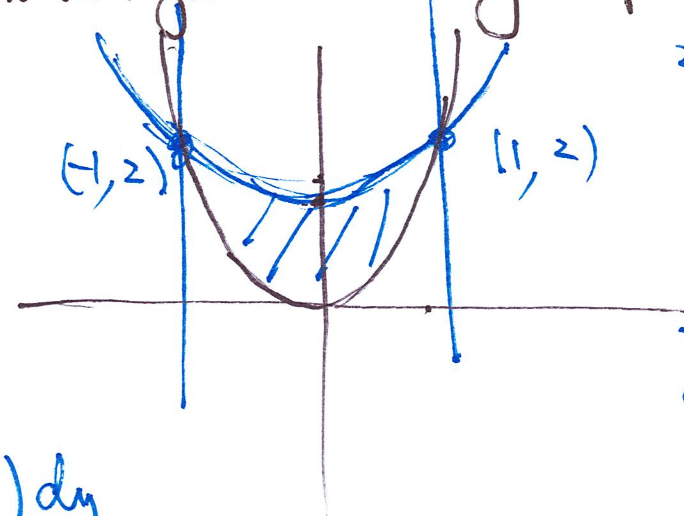
- D is type-II

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

examples

(1) $I = \iint_D (x+2y)\,dA$, D is the region bounded by the parabolas $y = 2x^2$ and $y = 1+x^2$





$2x^2 = 1+x^2$ $\qquad y-1 = x^2$

$x^2 = 1 \Rightarrow x = \pm 1$

$$\begin{cases} -1 \le x \le 1 \\ 2x^2 \le y \le 1+x^2 \end{cases}$$

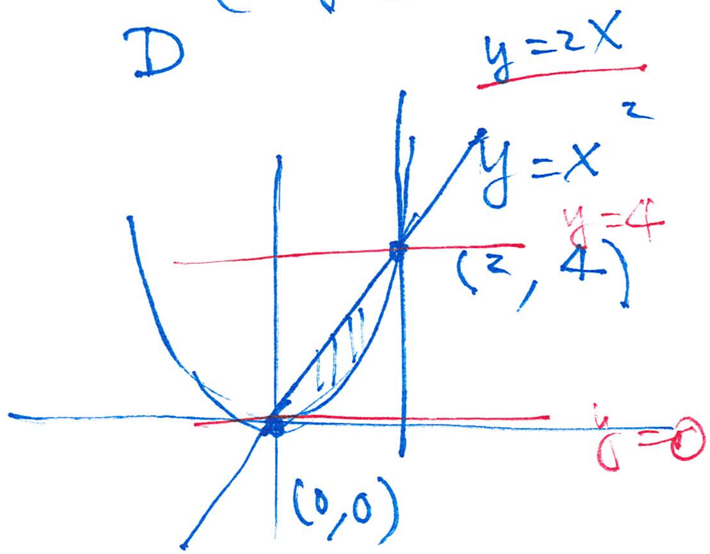
$$I = \int_{-1}^{1} dx \int_{2x^2}^{1+x^2} (x+2y)\,dy$$

(2) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

$$V = \iint_D (x^2 + y^2)\, dA$$

$$2x = x^2 \implies x(x-2) = 0$$

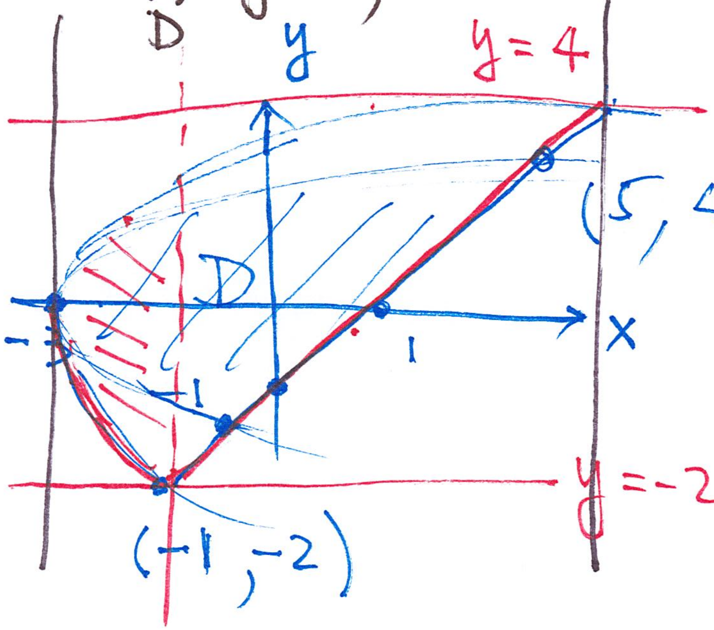
$$\implies x = 0 \text{ or } 2$$



$$\begin{cases} 0 \le x \le 2 \\ x^2 \le y \le 2x \end{cases}$$

$$\begin{cases} \frac{y}{2} \le x \le \sqrt{y} \\ 0 \le y \le 4 \end{cases}$$

(3) $\iint_D xy\, dA$, where D is the region bounded by the line $y = x-1$ and the parabola $y^2 = 2x+6$.



$x = y+1$ $\qquad$ $x+3 = \frac{1}{2}y^2$

$y^2 = 2x+6 = 2(y+1)+6 = 2y+8$

$y^2 - 2y + 1 = 9$ $\qquad$ $(y-1)^2 = 9 \Rightarrow y = 1 \pm 3$

$= -2,\ 4$

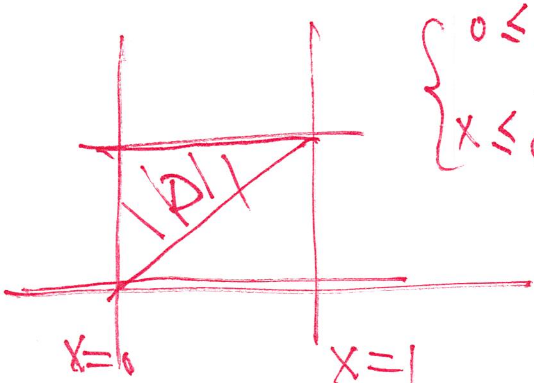
$$\begin{cases} -2 \le y \le 4 \\ \frac{1}{2}y^2 - 3 \le x \le y+1 \end{cases}$$

$$\begin{cases} -3 \le x \le \cancel{5}\ -1 \\ -\sqrt{2x+6} \le y \le \sqrt{2x+6} \end{cases}$$

(4) Find the volume of the tetrahedron bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$, and $z=0$.

$$\begin{cases} \cancel{-1} \le x \le 5 \\ x-1 \le y \le \sqrt{2x+6} \end{cases}$$

(5) Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \int_0^1 dy \int_0^y \underline{\sin y^2}\,dx$



$\begin{cases} 0 \le x \le 1 \\ x \le y \le 1 \end{cases}$ $\qquad \begin{cases} 0 \le y \le 1 \\ 0 \le x \le y \end{cases}$

$= \int_0^1 \underline{y} \sin y^2 \, dy$

$= -\frac{1}{2} \cos y^2 \Big|_0^1$

- <u>Properties of Double Integrals</u>

(1) $\iint_D (f+g)\,dA = \iint_D f\,dA + \iint_D g\,dA$; (2) $\iint_D c\,f(x,y)\,dA = c \iint_D f\,dA$

(3) $f(x,y) \ge g(x,y) \ \forall (x,y) \in D \implies \iint_D f\,dA \ge \iint_D g\,dA$

(4) $\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$

$D = D_1 \cup D_2$

$D_1 \cap D_2 = \phi$

(5) $\iint_D 1\,dA = A(D)$

(6) $m \le f(x,y) \le M \ \forall (x,y) \in D \implies m \le \frac{1}{A(D)} \iint_D f\,dA \le M$.