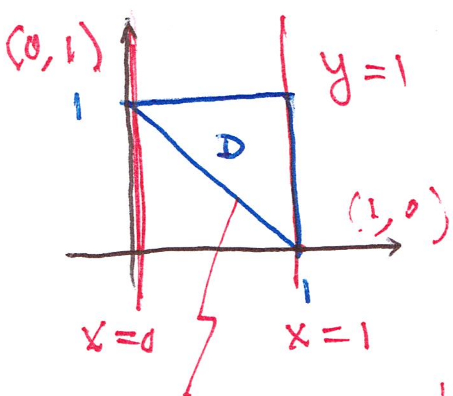
§15.4 Applications of Double Integrals

- density and mass (the charge density and electric charge)

$$m = \iint_D \rho(x,y)\,dA$$

Ex. 1 Charge is distributed over the region D so that the charge density at (x,y) is $\sigma(x,y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.



$x+y=1$

$$\frac{y-0}{x-1} = \frac{1-0}{0-1}$$

$$y = 1-x$$

$$\iint_D \sigma(x,y)\,dA = \int_0^1 \left(\int_{1-x}^1 xy\,dy \right) dx$$

$$= \int_0^1 \frac{1}{2}xy^2 \Big|_{y=1-x}^{1} dx = \int_0^1 \frac{1}{2}x(1-x)^2\,dx$$

$$\frac{1}{2}x(1-2x+x^2)$$

- <u>moments and centers of mass</u>

<u>moments</u>

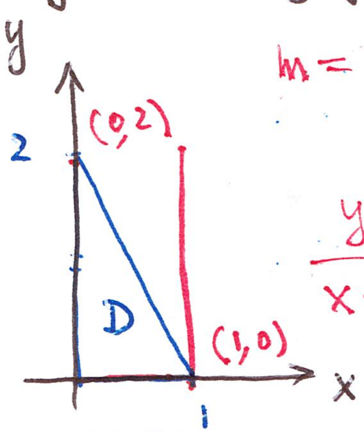
$$M_x = \iint_D y\,\rho(x,y)\,dA, \qquad \text{about the } x\text{-axis}$$

$$M_y = \iint_D x\,\rho(x,y)\,dA, \qquad \text{about the } y\text{-axis}$$

<u>the center of mass</u>

$$(\bar{x}, \bar{y}) = \frac{1}{m}(M_y, M_x)$$

<u>Ex. 2</u> Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, and $(0,2)$ if the density function is $\rho(x,y) = 1+3x+y$.



$0 \le x \le 1$

$0 \le y \le 2(1-x)$

$$m = \iint_D \rho\,dA = \int_0^1 \int_0^{2(1-x)} (1+3x+y)\,dy\,dx = \int_0^1 (1+3x)y + \frac{1}{2}y^2 \Big|_{y=0}^{2(1-x)} dx$$

$$\frac{y}{x-1} = \frac{2}{-1}$$

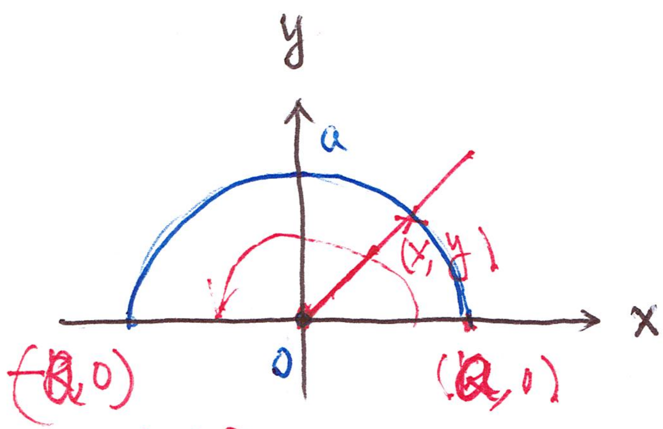
$$= \int_0^1 \left[(1+3x)\cdot 2(1-x) + \frac{1}{2}(2(1-x))^2\right] dx$$

$$M_y = \iint_D x(1+3x+y)\,dA, \quad M_x = \iint_D y(1+3x+y)\,dA$$

Ex. 3 The density at any point on a ~~semicircle~~ semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

$\rho(x,y) = k\sqrt{x^2+y^2}$

$(\bar{x}, \bar{y}) = \frac{1}{m}\left(M_y, M_x\right)$



D: $0 \le \theta \le \pi$

$0 \le r \le a$

$$m = \iint_D \rho(x,y)\,dA = k\iint_D \sqrt{x^2+y^2}\,dA$$

$$= k\int_0^{\pi} d\theta \int_0^a r\,dr \cdot r = k\pi \frac{1}{3} r^3 \Big|_0^a$$

$$= \frac{1}{3} k\pi a^3$$

$$M_y = \iint_D x\,\rho(x,y)\,dA$$

$$= k\int_0^{\pi} d\theta \int_0^a r\cos\theta \cdot r\; r\,dr$$

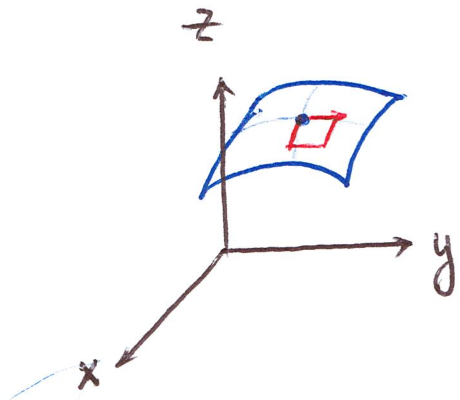
$$= k\int_0^{\pi} \cos\theta\,d\theta \int_0^a r^3\,dr = k\Big[\sin\theta\Big]_0^{\pi} \frac{1}{4} r^4\Big|_0^a = 0$$

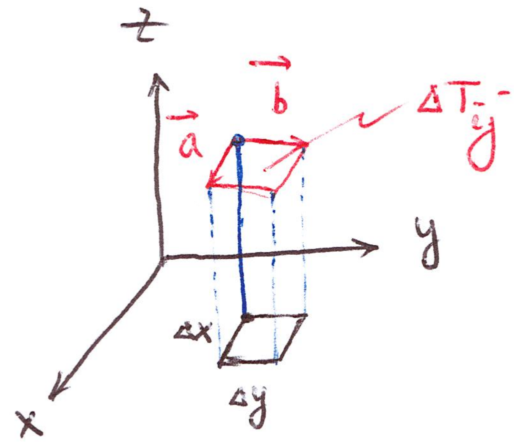
$$M_x = k\iint_D y\,\rho\,dA = k\int_0^{\pi} d\theta \int_0^a r\sin\theta \cdot r \cdot r\,dr = k\Big[-\cos\theta\Big]_0^{\pi} \frac{1}{4} r^4\Big|_0^a = \frac{1}{2} k a^4$$

$$(\bar{x}, \bar{y}) = \frac{\left(0, \frac{1}{2} k a^4\right)}{\frac{1}{3} k\pi a^3} = \left(0, \frac{3a}{2\pi}\right)$$

§15.5 Surface Area

__surface S__ : $z = f(x,y)$, $(x,y) \in D$





$$\vec{a} = \langle \Delta x,\ 0,\ \frac{\partial f}{\partial x}(x_i, y_j)\,\Delta x \rangle$$

$$\vec{b} = \langle 0,\ \Delta y,\ \frac{\partial f}{\partial y}(x_i, y_j)\,\Delta y \rangle$$

$$\Delta T_{ij} = |\vec{a} \times \vec{b}|$$

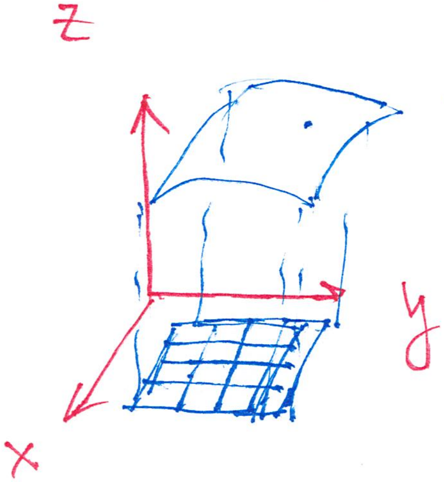
$$= \sqrt{1 + \frac{\partial f}{\partial x}(x_i, y_j)^2 + \frac{\partial f}{\partial y}(x_i, y_j)^2}\ \Delta x \Delta y$$

$$A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

$$= \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ dA$$

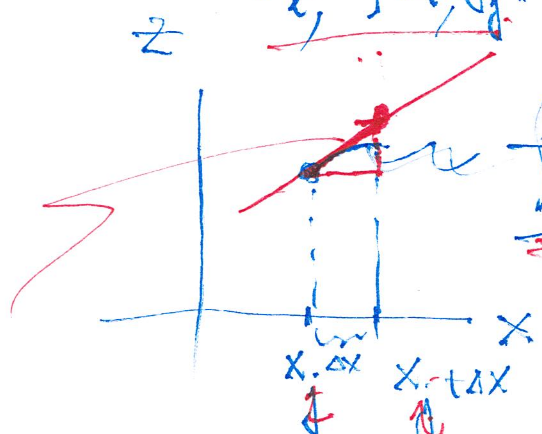
§15.5 Surface Area

$$z = f(x,y),\quad (x,y)\in D$$

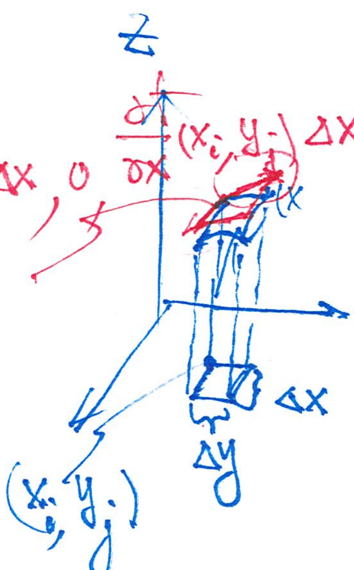


$$S = \{(x,y,f(x,y)) \mid (x,y)\in D\}$$

$$z = f(x_i,y_j) + \frac{\partial f}{\partial x}(x_i,y_j)(x - x_i)$$



$$\left(x_i+\Delta x,\ f(x_i,y_j)+\frac{\partial f}{\partial x}(x_i,y_j)\Delta x\right)$$

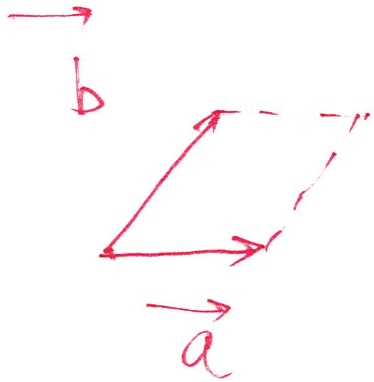


$$\vec a = \left(\Delta x,\ 0,\ \frac{\partial f}{\partial x}(x_i,y_j)\Delta x\right)$$

$$\vec b = \left(0,\ \Delta y,\ \frac{\partial f}{\partial y}(x_i,y_j)\Delta y\right)$$

$$\left(\Delta x,\ \frac{\partial f}{\partial x}(x_i,y_j)\Delta x\right)$$

$$\Delta x \Delta y$$



$$\left|\vec{a}\times\vec{b}\right| = \cancel{\text{abs}}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & \frac{\partial f}{\partial x}(x_i, y_j)\,\Delta x \\ 0 & \Delta y & \frac{\partial f}{\partial y}\,\Delta y \end{vmatrix}$$

$$= \left|\left(-\frac{\partial f}{\partial x}\Delta x\Delta y,\ -\frac{\partial f}{\partial y}\Delta x\Delta y,\ \Delta x\Delta y\right)\right|$$

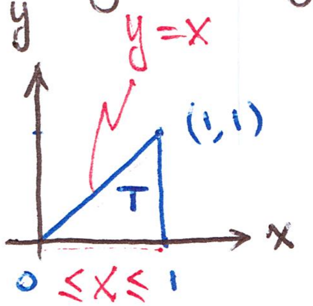
$$= \left|\left(-\frac{\partial f}{\partial x},\ -\frac{\partial f}{\partial y},\ 1\right)\right|\Delta x\Delta y$$

$$= \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\ \Delta x\Delta y$$

$$A(S) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{i,j} \sqrt{1 + \left(\frac{\partial f}{\partial x}(x_i, y_j)\right)^2 + \left(\frac{\partial f}{\partial y}(x_i, y_j)\right)^2}\ \Delta x \Delta y$$

$$A(S) = \iint_D \underbrace{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ dA}_{dS}$$

Ex. 1 Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vercies (0,0), (1,0), and (1,1).

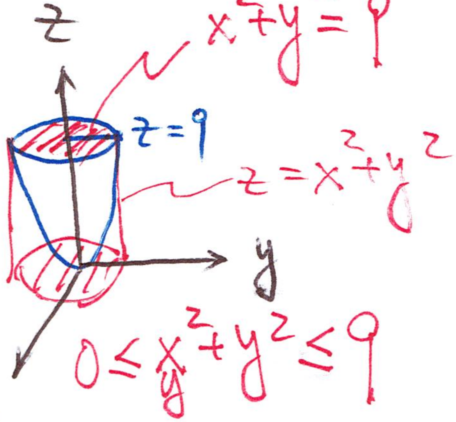


$$A(S) = \iint_T \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_T \sqrt{1+4x^2+4}\, dA$$

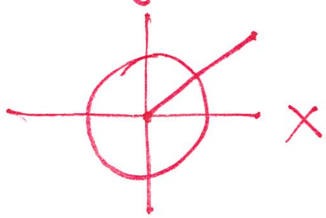
$$= \iint_T \sqrt{5+4x^2}\, dA = \int_0^1 \left(\int_0^x \sqrt{5+4x^2}\, dy\right) dx$$

$$= \int_0^1 x\sqrt{5+4x^2}\, dx \;\xrightarrow[du = 8x\,dx]{u=5+4x^2:\ 5\to 9}\; \int_5^9 \frac{1}{8} u^{\frac{1}{2}}\, du = \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_5^9$$

Ex. 2 Find the area of the part of the paraboloid $z = x^2+y^2$ that lies under the plane $z = 9$.



D: $0 \le x^2+y^2 \le 9$



$$A(S) = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1+(2x)^2+(2y)^2}\, dA$$

$$= \int_0^3 \int_0^{2\pi} \sqrt{1+4r^2}\, d\theta\, r\,dr = 2\pi \int_0^3 r\sqrt{1+4r^2}\, dr$$

$$\xrightarrow[du = 8r\,dr]{u=1+4r^2:\ 1\to 37}\; \int_1^{37} \frac{1}{8} u^{\frac{1}{2}}\, du$$