§15.7 Triple Integrals

Def.

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$

where $B = [a,b]\times[c,d]\times[r,s]$

Fubini's Thrm f is continuous on the rectangular box $B=[a,b]\times[c,d]\times[r,s]$

$$\iiint_B f(x,y,z)\,dV = \int_r^s \left[\int_c^d \left(\int_a^b f(x,y,z)\,dx\right)dy\right]dz$$

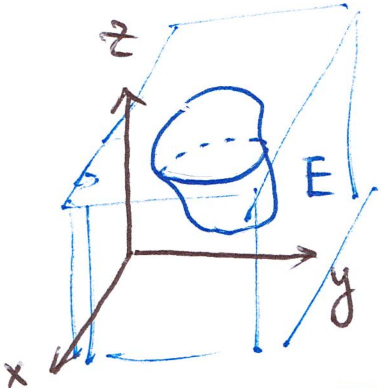
Ex. 1 $\iiint_B xyz^2\,dV$, $B=[0,1]\times[-1,2]\times[0,3]$

$$= \left(\int_0^1 x\,dx\right)\left(\int_{-1}^2 y\,dy\right)\left(\int_0^3 z^2\,dz\right)$$

$$= \int_0^3\left[\int_{-1}^2\left(\int_0^1 xyz^2\,dx\right)dy\right]dz = \int_0^3\int_{-1}^2 \frac{1}{2}x^2yz^2\Big|_{x=0}^{1}\,dy = \frac{1}{2}\int_0^3 \frac{1}{2}y^2\Big|_{y=-1}^{2} z^2\,dz$$

$$= \frac{3}{4}\int_0^3 z^2\,dz = \frac{1}{4}z^3\Big|_0^3 = \frac{27}{4}$$

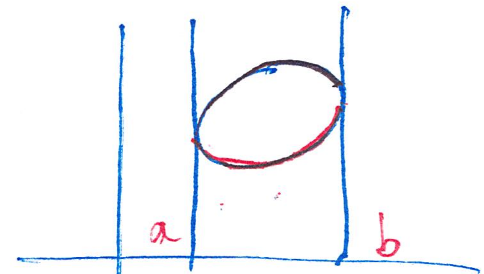
- **general region**



cube $R \supset E$

$$\iiint_E f(x,y,z)\,dV = \iiint_R \hat{f}\,dV$$

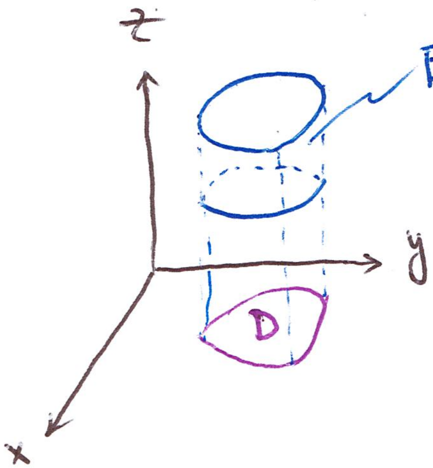
$$\hat{f} = \begin{cases} f & (x,y,z) \in E \\ 0 & \text{otherwise} \end{cases}$$



$a \le x \le b$

- **elementary regions**

$$\iiint_E f\,dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f\,dz\,dA$$



$$E: \begin{cases} u_1(x,y) \le z \le u_2(x,y) \\ (x,y) \in D \end{cases}$$

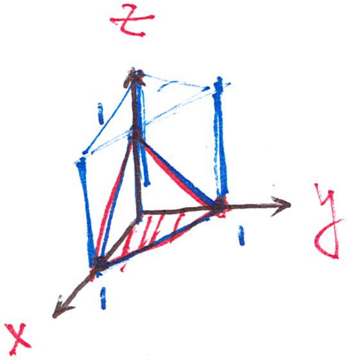
type-I

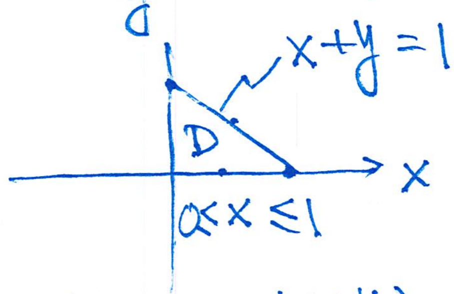
type-II $\begin{cases} u_1(x,z) \le y \le u_2(x,z) \\ (x,z) \in D \end{cases}$

type-III $\begin{cases} u_1(y,z) \le x \le u_2(y,z) \\ (y,z) \in D \end{cases}$

examples

(1) Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$, and $x+y+z=1$.





$$0 \le z \le 1-(x+y)$$

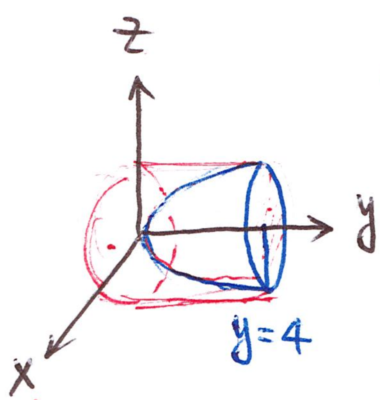
$$I = \iint_D \int_0^{1-(x+y)} z\,dz\,dA$$

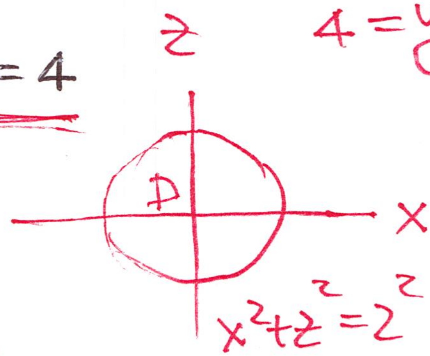
$$= \iint_D \frac{1}{2}\left[1-(x+y)\right]^2 dA$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (1-x-y)^2\,dy\,dx$$

(2) Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the region bounded by the paraboloid $y=x^2+z^2$ and the plane $y=4$

$$4 = y = x^2+z^2$$





$$I = \iint_D \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\,dA$$

$$= \iint_D \left[4-(x^2+z^2)\right](x^2+z^2)^{\frac{1}{2}}\,dA$$

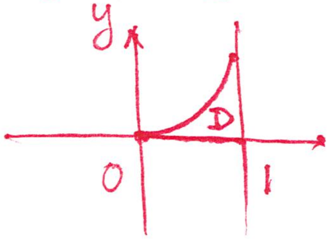
$$= \int_0^{2\pi} d\theta \int_0^2 (4-r^2)\,r\cdot r\,dr = 2\pi\int_0^2 (4r^2-r^4)\,dr$$

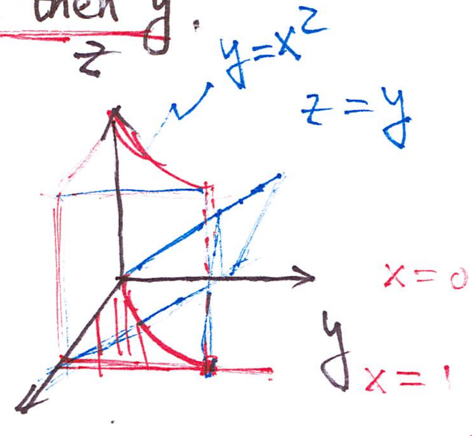
$$= 2\pi\left[\frac{4}{3}r^3 - \frac{1}{5}r^5\right]_0^2 =$$

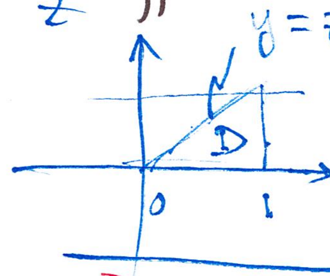
(3) Express the iterate integral $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx$ as a triple integral and then rewrite it as an iterate integral in a different order, integrating first with respect to x, then z, and then y.

$0 \le z \le y$

$\begin{cases} 0 \le y \le x^2 \\ 0 \le x \le 1 \end{cases}$



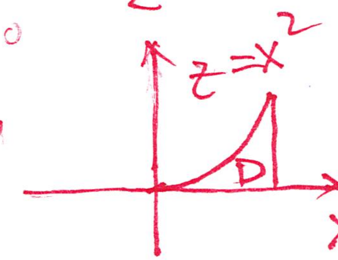




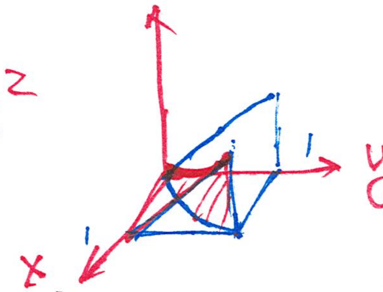
$(y,z) \in D$

$\begin{cases} 0 \le z \le 1 \\ z \le y \le 1 \end{cases}$

$\sqrt{y} \le x \le 1$



$0 \le y \le x^2$



(4) Find the volume of the solid enclosed by the paraboloid $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.

$$V = \iiint_E 1\,dV$$

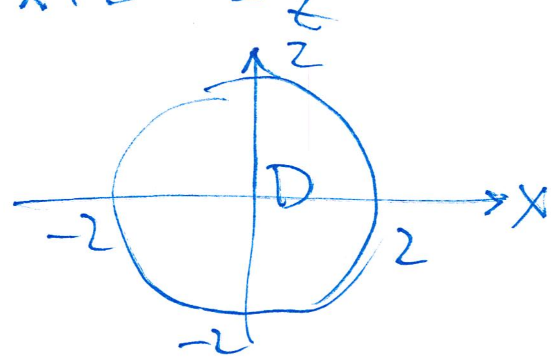
$$= \iint_D \int_{x^2+z^2}^{8-x^2-z^2} dy\,dA$$

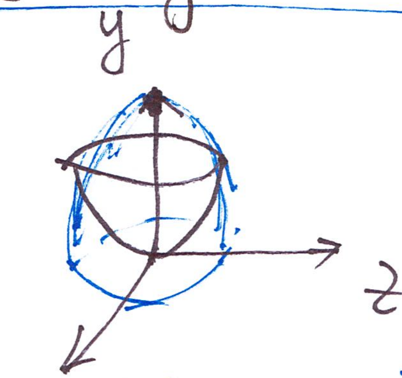
$$= \iint_D 2(4 - x^2 - z^2)\,dA$$

$$= 2\int_0^{2\pi} d\theta \int_0^2 (4 - r^2)\cdot r\,dr$$

$x^2 + z^2 = y = 8 - x^2 - z^2$

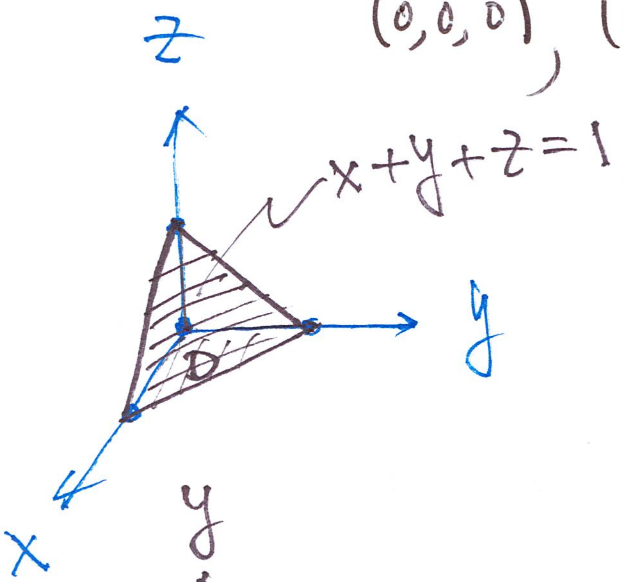
$x^2 + z^2 = 2^2$





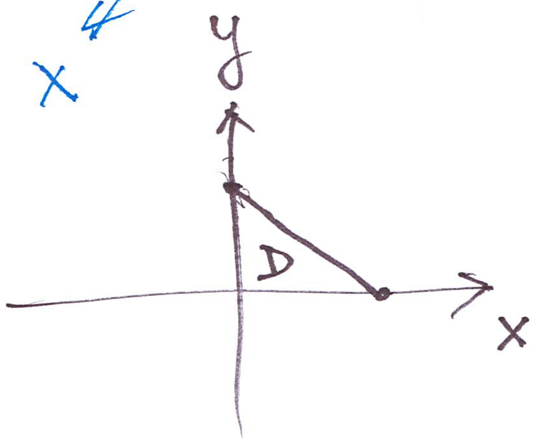
$x^2 + z^2 \le y \le 8 - x^2 - z^2$

#16 (P1038) $I = \iiint_T xz\,dV$, where T is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$

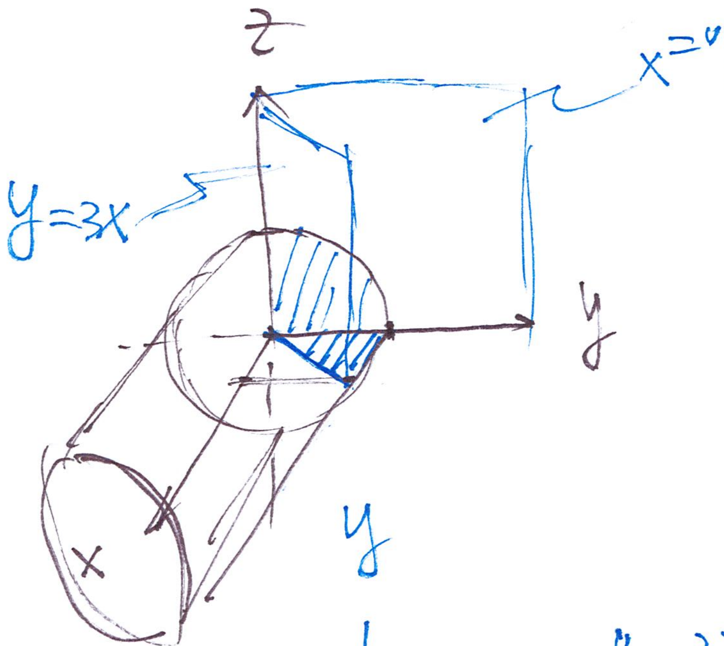


$$I = \iint_D \int_0^{1-x-y} xz\,dz\,dA$$

$$= \iint_D x\,\frac{1}{2}(1-x-y)^2\,dA$$



#18 (P1038) $\iiint_E z\,dV$, where E is bounded by the cylinder $y^2+z^2=9$ and the planes $x=0$, $y=3x$, and $z=0$ in the first octant.



$$\iiint_E z\,dV = \iint_D \int_0^{\sqrt{9-y^2}} z\,dz\,dA$$

$$= \iint_D \frac{1}{2}(9-y^2)\,dA$$

$$=$$