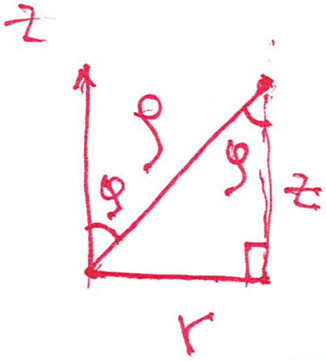
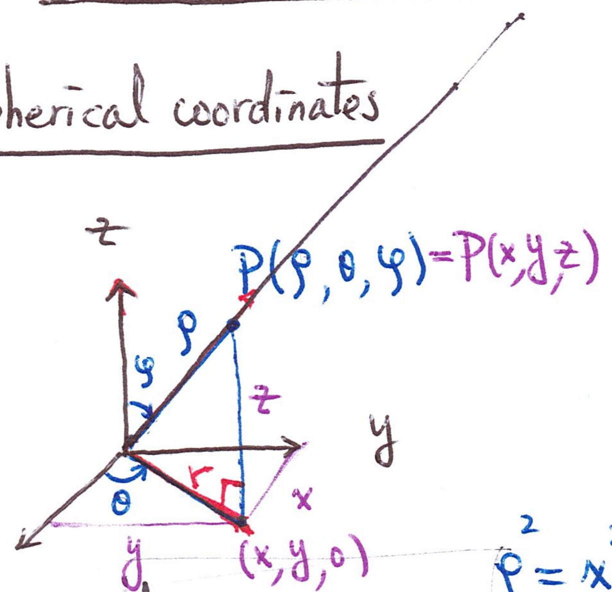
§15.8 Triple Integrals in Spherical Coordinates



- spherical coordinates



$0 \le \rho \le +\infty$

$$\begin{cases} x = (\rho \sin\varphi)\cos\theta \\ y = (\rho \sin\varphi)\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

($r = \rho\sin\varphi$)

$\rho \ge 0$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

$\sin\varphi = \frac{r}{\rho}$

$\Rightarrow r = \rho\sin\varphi$

$\rho^2 = x^2 + y^2 + z^2$

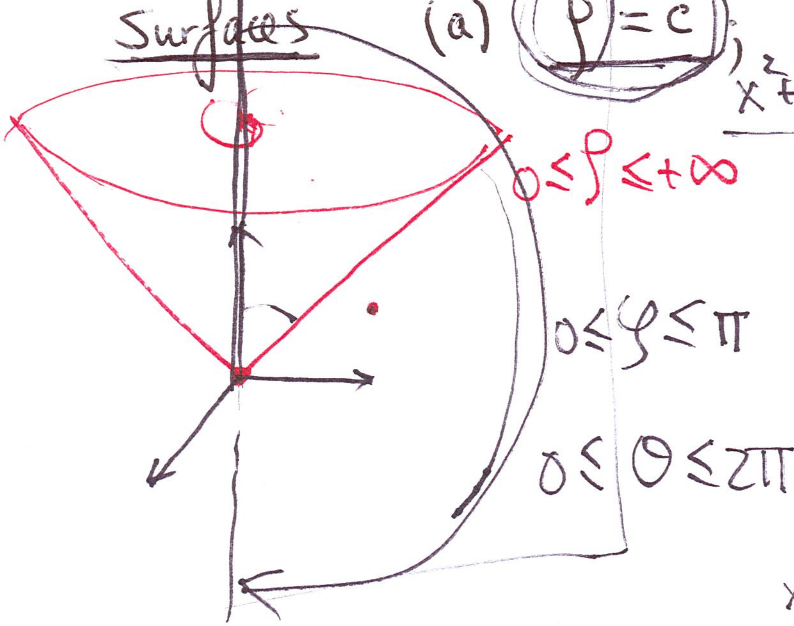
$\tan\theta = \frac{y}{x}$, $\varphi = \arccos\frac{z}{\rho}$

$\theta = \tan^{-1}\frac{y}{x}$

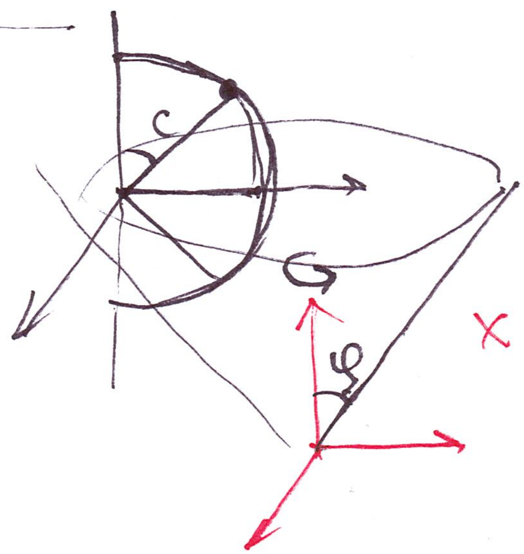
Surfaces (a) $\rho = c$ ($\rho^2 = c^2$); (b) $\theta = c$; (c) $\varphi = c$.

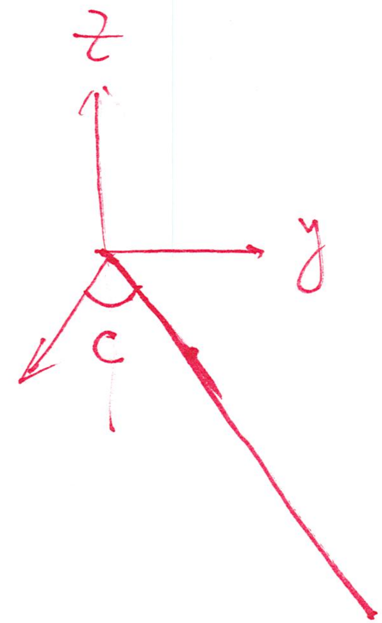
c – constant

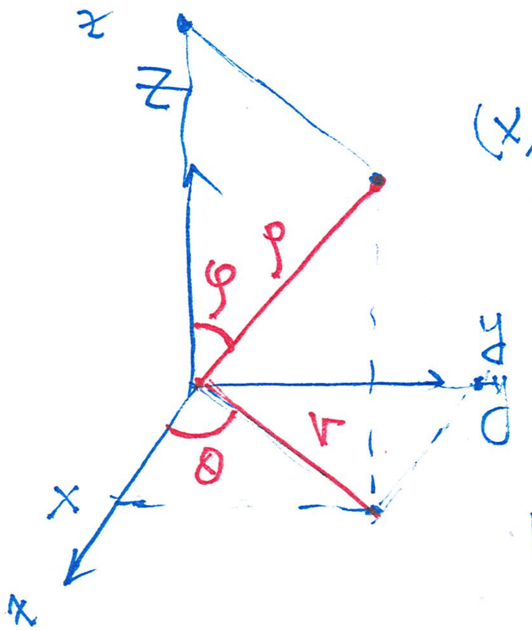
$x^2 + y^2 + z^2 = c^2$

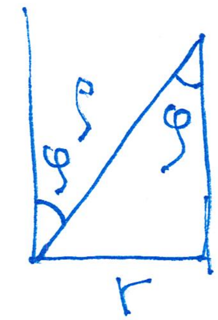


$x^2 +$









$r = \rho\sin\varphi$

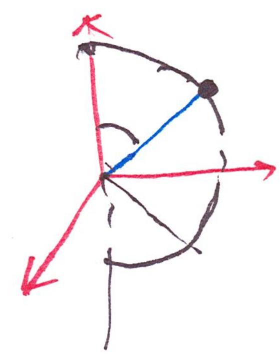
$$x = r\cos\theta = \rho\cos\theta\sin\varphi$$

$$y = r\sin\theta = \rho\sin\theta\sin\varphi$$

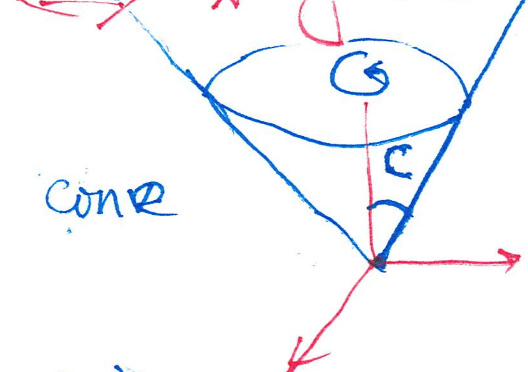
$$z = \rho\cos\varphi$$

$$0 \le \rho < +\infty, \quad 0 \le \varphi \le \pi, \quad 0 \le \theta \le 2\pi$$

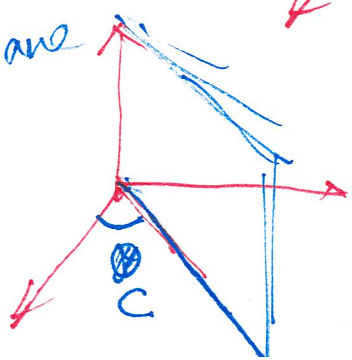
(a) $\rho = C \iff x^2+y^2+z^2 = C$ sphere



(b) $\varphi = C$ cone



(c) $\theta = C$ plane

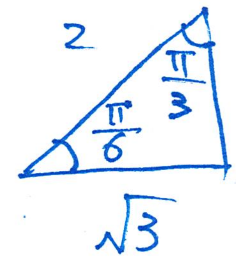


Ex. 1 Find its rectangular coordinates of the point $(2, \frac{\pi}{4}, \frac{\pi}{3})$ given in spherical coordinates. $= (\rho, \theta, \varphi)$

$$x = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = 2\,\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$

$$y = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$

$$z = 2\cos\frac{\pi}{3} = 2\,\frac{1}{2}$$



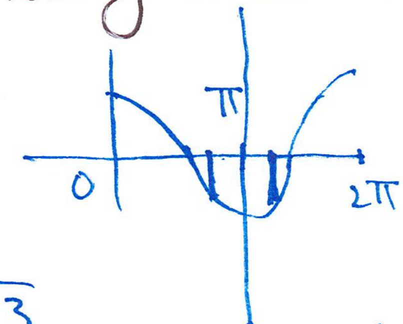
Ex. 2 Find the spherical coordinates of the point $(0, 2\sqrt{3}, -2)$ given in rectangular coordinates. $= (x, y, z)$

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0+12+4} = 4$$

$$\cos\varphi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \Rightarrow \varphi = \frac{2\pi}{3}$$

$$= \frac{2\sqrt{3}}{\sin\frac{2\pi}{3}} = \qquad \cos\theta = \frac{x}{\rho\sin\varphi} = 0$$

$$\theta = \frac{\pi}{2}$$



$$\left(4, \frac{\pi}{2}, \frac{2\pi}{3}\right)$$

• triple integral in spherical coordinates

$$\iiint_E f(x,y,z)\,dV = \iiint_E f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\underbrace{\boxed{\rho^2\sin\varphi}\,d\rho\,d\theta\,d\varphi}_{= dV}$$

$dA = r\,dr\,d\theta$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \text{abs.}\begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi} \end{vmatrix} = \rho^2\sin\varphi$$

Ex. 3 $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball, $B=\{(x,y,z) \mid x^2+y^2+z^2\le 1\}$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{\rho^3}\rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta$$

$0\le\rho\le 1$

$0\le\theta\le 2\pi$

$0\le\varphi\le\pi$

$$= 2\pi\left(\int_0^\pi \sin\varphi\, d\varphi\right)\left(\int_0^1 e^{\rho^3}\rho^2\, d\rho\right)$$

$$= 2\pi\left[-\cos\varphi\right]_0^\pi \frac13 e^{\rho^3}\Big|_0^1 = 2\pi\left[1-(-1)\right]\cdot\frac13(e-1)$$

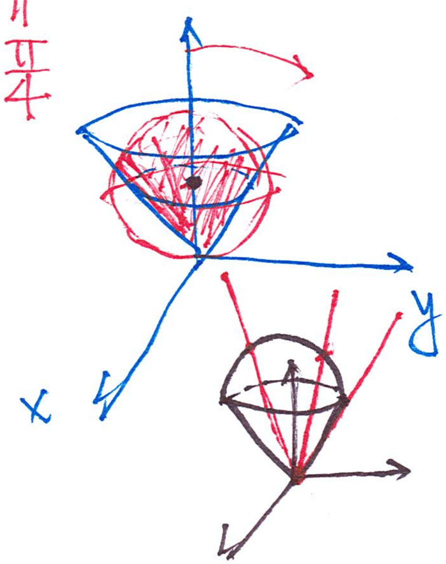
Ex. 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and ~~above~~ below the sphere $x^2+y^2+z^2=z$ $\rightarrow \rho=\cos\varphi$

$z^2 - z = z^2 - 2\cdot\frac12 z + \left(\frac12\right)^2 - \left(\frac12\right)^2$

$x^2+y^2+\left(z-\frac12\right)^2 = \left(\frac12\right)^2$

$D: \sqrt{x^2+y^2} = z = \frac12+\sqrt{\frac14-x^2-y^2}$

$\varphi = \frac{\pi}{4}$



$$V = \iint_D \int_{\sqrt{x^2+y^2}}^{\frac12+\sqrt{\frac14-x^2-y^2}} 1\, dz\, dA$$

$0\le\rho\le\cos\varphi$

$0\le\varphi\le\frac{\pi}{4}$

$0\le\theta\le 2\pi$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\left(\int_0^{\cos\varphi} 1\cdot\rho^2\sin\varphi\, d\rho\right) d\varphi\, d\theta$$

$$=2\pi\int_0^{\frac{\pi}{4}}\sin\varphi\, \frac13\rho^3\Big|_0^{\cos\varphi} d\varphi = \frac{2\pi}{3}\int_0^{\frac{\pi}{4}}\cos^3\varphi\sin\varphi\, d\varphi \xrightarrow[du=-\sin\varphi\, d\varphi]{u=\cos\varphi} -\frac{2\pi}{3}\int_1^{\frac{\sqrt2}{2}} u^3\, du$$

#40 Evaluate $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2z+y^2z+z^3)\,dV$ by using spherical coordinates.

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2\cdot\rho\,\cancel{\sin}\cos\varphi\;\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$-\sqrt{a^2-x^2-y^2} \le z \le \sqrt{a^2-x^2-y^2}$

$-\sqrt{a^2-y^2} \le x \le \sqrt{a^2-y^2}$

$-a \le y \le a$

$z=-\sqrt{a^2-x^2-y^2}$

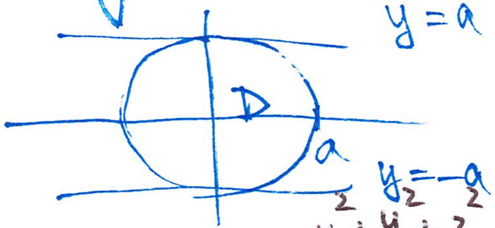
$z=\sqrt{a^2-x^2-y^2}$

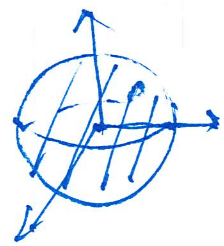
$0\le\rho\le a$

$0\le\varphi\le\pi$

$0\le\theta\le 2\pi$

$$= 2\pi\int_0^{\pi}\cos\varphi\sin\varphi\,d\varphi\int_0^a \rho^5\,d\rho$$





#25 $\displaystyle\iiint_E x\,e^{x^2+y^2+z^2}\,dV$, E: the portion of the unit ball $x^2+y^2+z^2\le 1$ that lies in the 1st octant.

$$= \int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^1 \rho\sin\varphi\cos\theta\,e^{\rho^2}\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$0\le\rho\le 1$

$0\le\varphi\le\frac{\pi}{2}$

$0\le\theta\le\frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\cdot\int_0^{\frac{\pi}{2}}\sin^2\varphi\,d\varphi\int_0^1 \rho^3 e^{\rho^2}\,d\rho = \int_0^1 \frac{1}{2}u\,e^u\,du$$

$\sin^2\varphi = \dfrac{1-\cos 2\varphi}{2}$

$u=\rho^2$

$du = 2\rho\,d\rho$