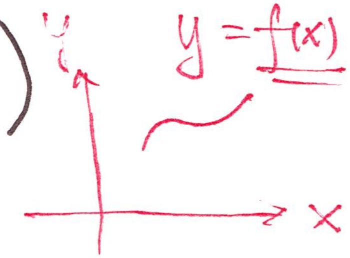
Chapter 16 Vector Calculus (12 lectures)



$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

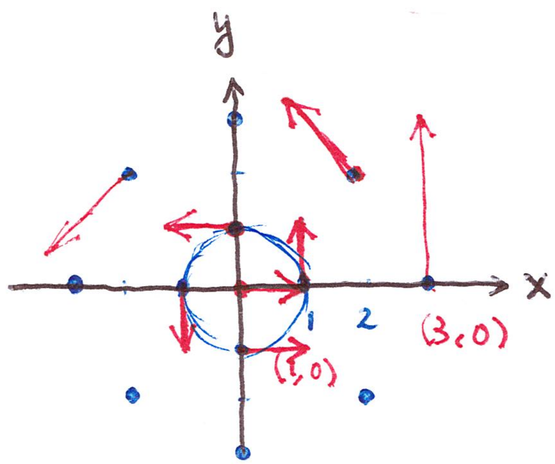
§16.1 Vecto Fields

vector fields

$\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} = \langle P(x,y), Q(x,y) \rangle : R^2 \to R^2$

$\vec{F}(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle : R^3 \to R^3$

Ex. 1 Describe $\vec{F}(x,y) = \langle \underbrace{-y}_{P(x,y)}, \underbrace{x}_{Q(x,y)} \rangle$ by sketching some of the vectors $\vec{F}(x,y)$ as in Fig. 3.



$\vec{F}(1,0) = \langle 0, 1 \rangle$

$\vec{F}(0,1) = \langle -1, 0 \rangle$

$\vec{F}(-1,0) = \langle 0, -1 \rangle$

$\vec{F}(0,-1) = \langle 1, 0 \rangle$

$\vec{F}(3,0) = \langle 0, 3 \rangle$

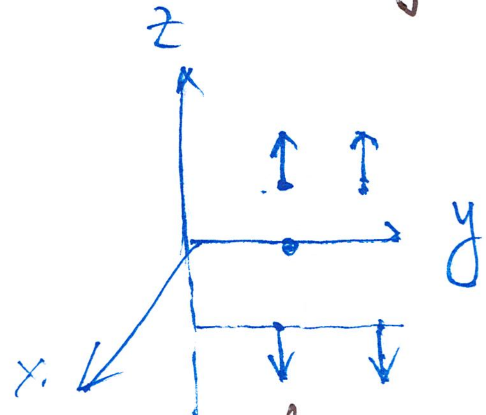
$\vec{F}(2,2) = \langle -2, 2 \rangle = 2\langle -1, 1 \rangle$

$\vec{F}(-2,2) = \langle -2, -2 \rangle = -2\langle 1, 1 \rangle$

$\vec{x} \cdot \vec{F}(\vec{x}) = \langle x, y \rangle \cdot \langle -y, x \rangle = -xy + yx = 0$

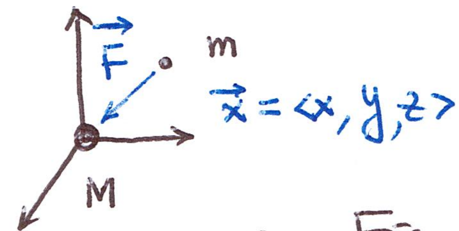
$|\vec{F}(x,y)| =$

Ex. 2 Sketch the vector field on R^3 given by $\vec{F}(x,y,z) = \langle 0, 0, z \rangle$.



Ex. 4 Gravitational Force

$$\vec{F}(\vec{x}) = -\frac{mMG}{|\vec{x}|^3}\vec{x}$$



see Fig. 14

$$= \nabla f \quad , \quad \text{where } f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$$

- gradient fields

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Ex.6 Find the gradient vector field of $f(x,y) = x^2 y - y^3$.
Plot the gradient field together with a contour map of f (see Fig. 15).
How are they related?

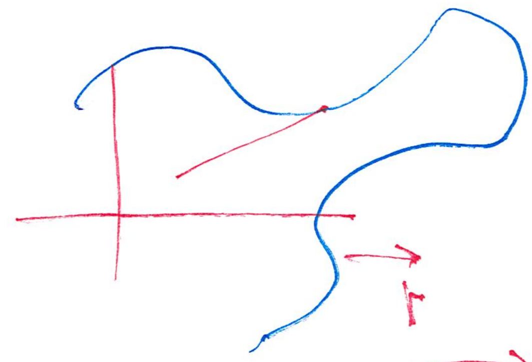
$f(x,y) = x^2 y - y^3 = c$

$\nabla f = \langle 2xy, x^2 - 3y^2 \rangle$

Def. $\vec{F}$ is a conservative field $\iff \exists f$ s.t. $\vec{F} = \nabla f$

§16.2 Line Integrals

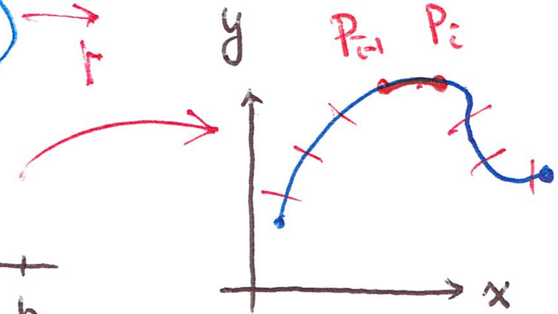
- scalar function
- vector field



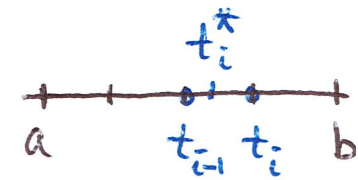
- line integral of scalar function

a plane curve C : $\vec{r}(t) = \langle x(t), y(t) \rangle$

$a \le t \le b$



Assume that C is a smooth curve. $\left(\vec{r}\,' \text{ is cont. and } \vec{r}\,'(t) \ne \vec{0}\right)$



Partition of $[a,b]$: $a = t_0 < t_1 < t_2 \cdots < t_n = b$

$$t_i = a + i\Delta t, \quad \Delta t = \frac{b-a}{n}$$

definition the line integral of f along C

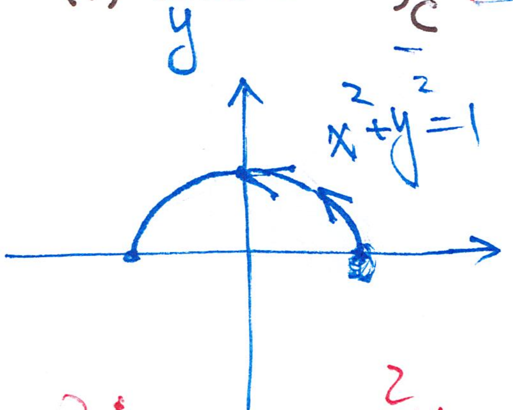
ds

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

$$= \int_a^b f(x(t), y(t)) \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt} = \int_a^b f(\vec{r}(t))\,\underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$$

Examples

(1) Evaluate $\underset{\text{II}}{I} = \int_C (2+x^2 y)\,ds$, where C is the upper half of the unit circle $x^2+y^2=1$



$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $\quad 0 \le t \le \pi$

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$$I = \int_0^{\pi} \left(2 + \cos^2 t \sin t\right) \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

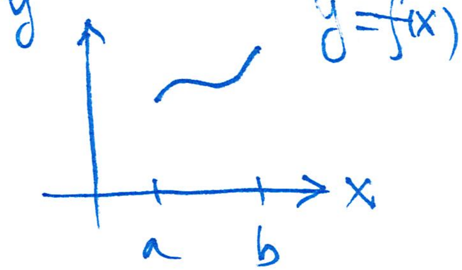
$f(x,y) = 2 + x^2 y$

$u = \cos t$

$du = -\sin t\, dt$

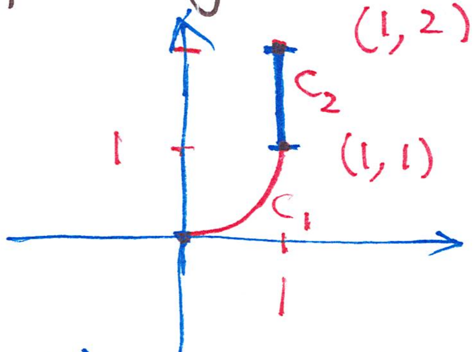
$$= \int_0^{\pi} (2 + \cos^2 t \sin t)\, dt = \left[2t\right]_0^{\pi} + \int_1^{-1} -u^2\, du$$

(2) Evaluate $I = \int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.



$\vec{r}(x) = \langle x, f(x) \rangle$

$x \in [a,b]$



$$I = \int_{C_1} + \int_{C_2} 2x\,ds$$

$$= \int_0^1 2\cdot t \left|\vec{r}_1{}'(t)\right| dt + \int_1^2 2\cdot 1 \left|\vec{r}_2{}'(t)\right| dt$$

$$= \int_0^1 2t \sqrt{1^2 + (2t)^2}\, dt + \int_1^2 2\cdot dt$$

$C_1 : \vec{r}_1(t) = \langle t, t^2 \rangle, \quad t \in [0,1]$

$C_2 : \vec{r}_2(t) = \langle 1, t \rangle, \quad t \in [1,2]$

$\dfrac{u = 1 + 4t^2}{du = 8t\,dt}$ $\quad \int_1^5 u^{\frac{1}{2}} \frac{1}{4}\, du + 2$