§16.3 The Fundamental Theorem for Line Integrals

$\vec{F}$ is conservative $\Longleftrightarrow \vec{F} = \nabla f$

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\| $$

$$\int_C \langle F', 0\rangle \cdot \vec{r}\,'dt = \int_C \langle F', 0\rangle \cdot d\vec{r}$$

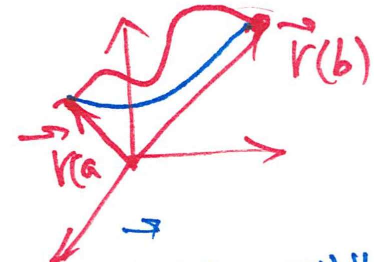
C : line segment $a \to b$

$\vec{r}(t) = \langle t, 0\rangle$

$a \le t \le b$

Theorem Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable function whose gradient vector ∇f is continuous on C.

$$\int_C \nabla f \cdot d\vec{r} = \boxed{f(\vec{r}(b)) - f(\vec{r}(a))}$$



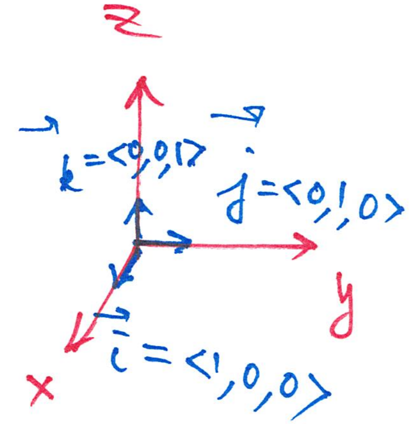
$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

$a \le t \le b$

Proof

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \underbrace{\boxed{\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}}_{\frac{d f(\vec{r}(t))}{dt}}\,dt = f(\vec{r}(t))\Big|_a^b$$

curl operator $\qquad \vec{F}(x,y,z) = \langle P(x,y,z), Q, R \rangle$

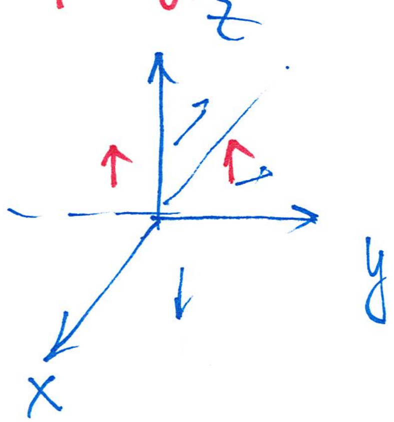


$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle \qquad \vec{F}(x,y,z) = \langle P, Q, 0 \rangle$

$$\nabla\times\vec{F}(x,y) = \nabla\times\langle P, Q, 0\rangle = \left\langle 0, 0, \boxed{\underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\nabla\times\vec{F}}} \right\rangle$$



Ex. 1 Find the work done by the gravitational field $\vec{F}(\vec{x}) = -\dfrac{mMG}{|\vec{x}|^3}\vec{x}$ in moving a particle with mass m from the point $(3,4,12)$ to the point $(2,2,0)$ along a piecewise smooth curve.

$\vec{x} = \langle x, y, z\rangle$

$$\vec{F} = \nabla f$$

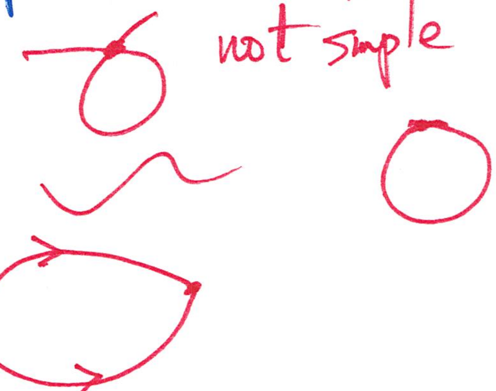
$$f = \frac{mMG}{|\vec{x}|} = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$$

$$W = \int_C \vec{F}\cdot d\vec{r} = f(2,2,0) - f(3,4,12)$$

$$= mMG\left[\frac{1}{\sqrt{2^2+2^2+0^2}} - \frac{1}{\sqrt{3^2+4^2+12^2}}\right]$$

$$\frac{1}{mMG}\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2x = -\frac{x}{|\vec{x}|^3}$$

Theorem Assume that $\vec{F} \in C^1(\mathbb{R}^3)^3$ and that C is a piecewise smooth curve. The following statements are equivalent.



(1) $\forall$ simple close curve C: $\int_C \vec{F}\cdot d\vec{r} = 0$.

$\Downarrow$

(2) $\int_C \vec{F}\cdot d\vec{r}$ is independent of path.

$\Downarrow$

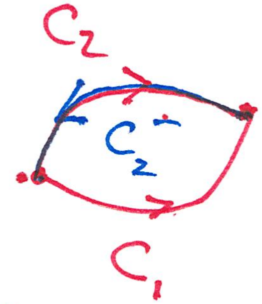
(3) there exists a function f such that $\vec{F} = \nabla f$.

$\Downarrow$

(4) $\nabla\times\vec{F} = \vec{0}$ on a simply connected domain.

$\nabla\times\vec{F} =$

Proof $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$



$(1) \Rightarrow (2)$

$\int_C \vec{F}\cdot d\vec{r}$ is indep. of path

$C_1 \cup C_2^-$

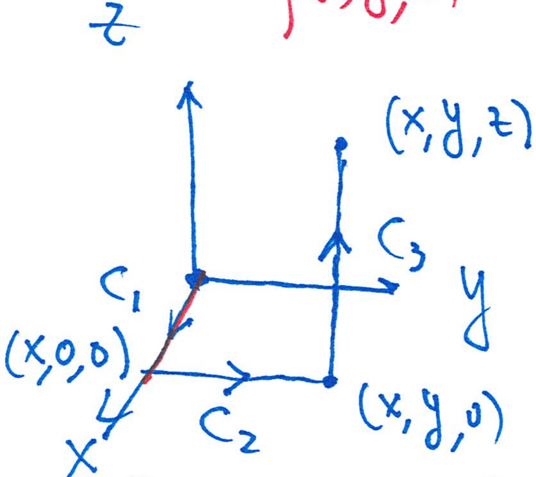
$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} \quad \Uparrow$$

$$\int_{C_1} + \int_{C_2^-} \vec{F}\cdot d\vec{r} = \int_{C_1 \cup C_2^-} \vec{F}\cdot d\vec{r} = 0$$

$$\| $$

$$\int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r}$$

$(2) \Rightarrow (3)$



$$f(x,y,z) = f(0,0,0) + \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$= f(0,0,0) + \int_0^x \vec{F}(t,0,0)\cdot \langle 1,0,0\rangle\, dt + \cdots$$

$$= f(0,0,0) + \int_0^x P(t,0,0)\, dt$$

$\vec{r}_1(t) = \langle t,0,0 \rangle, \ 0 \le t \le x$

$(3) \Rightarrow (4)$ $\quad \vec{F} = \nabla f \xRightarrow{?} \nabla \times \vec{F} = \vec{0}$

$\underline{(4) \Rightarrow (1)}$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = 0 \quad \text{(Stoke thm)}$$