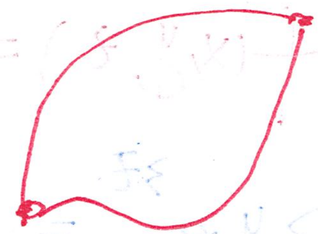
$\vec{F}$ is a conservative field ($\vec{F} = \nabla f$)

$$\iff \int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$



$\int_C \vec{F} \cdot d\vec{r}$ is path indep.

$$\iff \oint_C \vec{F} \cdot d\vec{r} = 0$$

$C \rightsquigarrow$ simple closed curve

$$\iff \boxed{\nabla \times \vec{F} = \vec{0}}$$

- curl of vector field

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle, \quad \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle, \quad \nabla \times \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Examples

$\nabla \times \vec{F} = \langle 1, -1 \rangle \neq \langle 0,0 \rangle \Rightarrow \vec{F}$ is not conservative

(1) Determine whether or not the vector field $\boxed{\vec{F}(x,y) = \langle x-y, x-2 \rangle}$ is conservative.

and $\vec{F}(x,y) = \langle 3+2xy, x^2-3y^2 \rangle$

If it is conservative, find f such that $\vec{F} = \nabla f$, and evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is a curve given by $\vec{r}(t) = \langle e^t \sin t, e^t \cos t \rangle$, $0 \le t \le \pi$.

$\vec{r}(\pi) = \langle 0, -e^{\pi} \rangle$, $\vec{r}(0) = \langle 0, e^{0} \rangle$

$$\nabla \times \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2x = 0$$

$f = ?$

$$\langle 3+2xy, x^2-3y^2 \rangle = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 3 + 2xy \Longrightarrow f(x,y) = 3x + x^2 y + C(y) \\ \dfrac{\partial f}{\partial y} = x^2 - 3y^2 = x^2 + C'(y) \Longrightarrow C'(y) = -3y^2 \Longrightarrow C(y) = -y^3 + D \end{cases}$$

$$f(x,y) = 3x + x^2 y - y^3 + D$$

$$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(\pi)) - f(\vec{r}(0))$$
$$= \left(-(-e^{\pi})^3 + D\right) - \left(-(1)^3 + D\right)$$
$$= \cancel{2e^{3\pi}}\ e^{3\pi} + 1$$

(2) Is $\vec{F} = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$ conservative?

If so, find f such that $\vec{F} = \nabla f$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & 2xy + e^{3z} & 3ye^{3z} \end{vmatrix} = \langle 3e^{3z} - 3e^{3z}, 0 - 0, 2y - 2y \rangle = \langle 0, 0, 0 \rangle$$

Find f s.t. $\vec{F} = \nabla f$

$$f(x,y,z) = xy^2 + ye^{3z} + D(z)$$

$$\Uparrow$$

Solution 1

$$\begin{cases} \dfrac{\partial f}{\partial x} = y^2 \Longrightarrow f(x,y,z) = xy^2 + C(y,z) \\ \dfrac{\partial f}{\partial y} = 2xy + e^{3z} = 2xy + \dfrac{\partial C}{\partial y} \Longrightarrow \dfrac{\partial C}{\partial y} = e^{3z} \Rightarrow C(y,z) = ye^{3z} + D(z) \\ \dfrac{\partial f}{\partial z} = 3ye^{3z} = 0 + 3ye^{3z} + D'(z) \Longrightarrow D'(z) = 0 \Rightarrow D(z) = \text{const.} \end{cases}$$

$$\boxed{f(x,y,z) = xy^2 + ye^{3z} + C}$$

Solution 2 $\quad \vec{F} = \langle P, Q, R \rangle$

$$f(x,y,z) = f(0,0,0) + \int_0^x P(x,0,0)\,dx + \int_0^y Q(x,y,0)\,dy + \int_0^z R(x,y,z)\,dz$$

$$= C + \int_0^x 0 \cdot dx + \int_0^y (2xy + 1)\,dy + \int_0^z 3ye^{3z}\,dz$$

$$= C + \left[xy^2 + y\right]_0^y + ye^{3z}\Big|_0^z = C + (xy^2 + y) + (ye^{3z} - y) = xy^2 + ye^{3z} + C$$

§16.4 Green's Theorem (2 dimensions)

vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

divergence $\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} \right\rangle \cdot \langle P, Q \rangle$

curl $\nabla \times \vec{F} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C (\vec{F} \cdot \vec{T})\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D \nabla \times \vec{F}\, dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

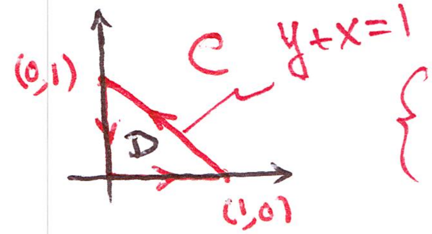
$C = \partial D^+$

$$\int_C (\vec{F} \cdot \vec{n})\, ds = \int_C P\,dy - Q\,dx = \iint_D \nabla \cdot \vec{F}\, dA$$

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

Examples

(1) Evaluate $\int_C x^4 dx + xy\, dy$, where



$$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le -x+1 \end{cases}$$

$$= \int_C \langle x^4, xy \rangle \cdot d\vec{r}$$

$$= \iint_D (\nabla \times \vec{F})\, dA = \iint_D (y-0)\, dA = \int_0^1 \int_0^{-x+1} y\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2 dx = \frac{1}{2} \cdot \frac{1}{3}(-1)(1-x)^3 \Big|_0^1 = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

(2) Evaluate $\oint_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4+1}\right) dy$, where C is the circle $x^2+y^2=9$.

counterclockwise

$$\begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases} \quad t: 0 \to 2\pi$$

$$= \int_C \langle 3y - e^{\sin x}, 7x + \sqrt{y^4+1} \rangle \cdot d\vec{r}$$

$$= \int_0^{2\pi} \langle 9\sin t - e^{\sin(3\cos t)}, 21\cos t + \sqrt{(3\sin t)^4+1} \rangle \cdot \langle -3\sin t, 3\cos t \rangle\, dt$$

$$= \iint_D (7-3)\, dA = 4\,|D| = 4 \cdot \pi \cdot 3^2 = 36\pi$$

- Area of D

$$A(D) = \iint_D \left(\frac{\partial x}{\partial x}\right) dA = \int_{C=\partial D^+} \langle 0, x\rangle \cdot d\vec{r} = \int_{\partial D^+} x\,dy$$

$$= \iint_D \frac{\partial y}{\partial y}\, dA = \int_{\partial D^+} \langle -y, 0\rangle \cdot d\vec{r} = \int_{\partial D^+} -y\,dx$$

$$\boxed{A(D) = \frac{1}{2}\int_{\partial D^+} -y\,dx + x\,dy}$$

Ex. 3 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ex. 4 Evaluate $\int_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$.