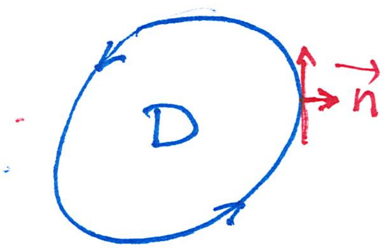
Green's Thrm (2 dimensions) $\vec{F} = \langle P(x,y), Q(x,y) \rangle$

(a) $$\int_C \vec{F} \cdot d\vec{r} = \int_C (\vec{F} \cdot \vec{T})\, ds = \iint_D \nabla \times \vec{F}\, dA$$



$C = \partial D^+$

(b) $$\int_C (\vec{F} \cdot \vec{n})\, ds = \iint_D \nabla \cdot \vec{F}\, dA$$

$$\nabla \times \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

(a) $$\Rightarrow A(D) = \int_{\partial D^+} x\, dy = -\int_{\partial D^+} y\, dx$$

$$= \frac{1}{2} \int_{\partial D^+} -y\, dx + x\, dy$$

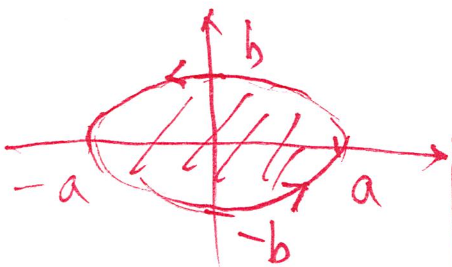
- <u>Area of D</u>

$$\boxed{A(D) = \iint_D \left(\frac{\partial x}{\partial x}\right) dA = \int_{C=\partial D^+} \langle 0, x\rangle \cdot d\vec{r} = \int_{\partial D^+} x\,dy}$$

$$= \iint_D \frac{\partial y}{\partial y}\, dA = \int_{\partial D^+} \langle -y, 0\rangle \cdot d\vec{r} = \int_{\partial D^+} -y\,dx$$

$$\boxed{= \frac{1}{2}\int_{\partial D^+} -y\,dx + x\,dy}$$

<u>Ex. 3</u> Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

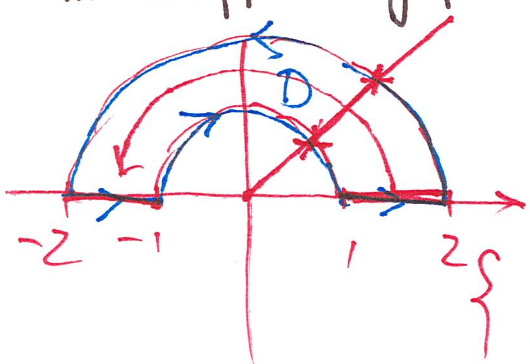


$$A(D) = \frac{1}{2}\int_{\partial D^+} -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} -\underline{b}\sin t \cdot \underline{a}(-\sin t)\,dt + \underline{a}\cos t \cdot \underline{b}\cos t\,dt$$

∂D^+: $\vec{r}(t) = \langle a\cos t, b\sin t\rangle$, $0 \le t \le 2\pi$

$$= \frac{1}{2}\int_0^{2\pi} ab\,dt = ab\pi$$

<u>Ex. 4</u> Evaluate $I = \int_{C^+} \underset{P}{y^2}\,dx + \underset{Q}{3xy}\,dy$, where C is the boundary of the semiannular region D in the upper half plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$.



$$\begin{cases} 1 \le r \le 2 \\ 0 \le \theta \le \pi \end{cases}$$

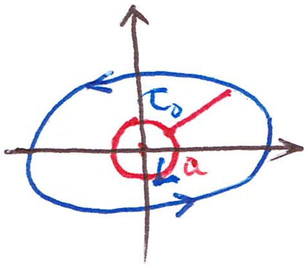
$$I = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (3y - 2y)\,dA = \int_1^2 \int_0^{\pi} r\sin\theta\,d\theta\,r\,dr$$

$$= \int_1^2 r^2\,dr \int_0^{\pi} \sin\theta\,d\theta$$

Ex. 5 $\vec{F}(x,y) = \left\langle -\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$,

Show that $\displaystyle\int_C \vec{F}\cdot d\vec{r} = 2\pi$ for every positively oriented simple closed path that encloses the origion.

$$\left[\begin{array}{l} \nabla\times\vec{F} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = \dfrac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2} + \dfrac{(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2} = 0 \\ \text{for } (x,y)\neq(0,0) \end{array}\right.$$



C

$$\int_{C\cup C^0} \vec{F}\cdot d\vec{r} = \iint_D \nabla\times\vec{F}\, dA = 0$$

$$\parallel$$

$$\int_C \vec{F}\cdot d\vec{r} + \int_{C^0} \vec{F}\cdot dr$$

$$\int_C \vec{F}\cdot d\vec{r} = -\int_{C^0} \vec{F}\cdot d\vec{r} = \underline{2\pi}$$

§16.5 Curl and Divergence

- $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

- <u>curl</u>

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

<u>Ex. 1</u> $\vec{F} = \langle xz, xyz, -y^2 \rangle$, $\nabla \times \vec{F} = \langle -2y - xy, x - 0, yz - 0 \rangle$

$$= \langle -2y - xy, x, yz \rangle$$

Theorem Assume that $f(x,y,z)$ has continuous second-order partial derivatives

$$\Longrightarrow \quad \operatorname{curl}(\nabla f) = \vec{0} \qquad \nabla f = \left\langle \underset{\overset{\parallel}{P}}{\frac{\partial f}{\partial x}}, \underset{\overset{\parallel}{Q}}{\frac{\partial f}{\partial y}}, \underset{\overset{\parallel}{R}}{\frac{\partial f}{\partial z}} \right\rangle$$

$$\left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \text{''}, \text{''} \right\rangle = \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, *, * \right\rangle = \langle 0,0,0 \rangle$$

- $\vec{F}$ is conservative $\Longleftrightarrow$ $\vec{F} = \nabla f$.

Ex. 2 Show that the vector field $\vec{F} = \langle xz, xyz, -y^2 \rangle$ is not conservative.

$\vec{F}$ is conservative $\Longrightarrow \vec{F} = \nabla f \Longrightarrow \nabla \times \vec{F} = \vec{0}$

$\nabla \times \vec{F} = \langle -2y - xy, x, yz \rangle \neq \langle 0, 0 \rangle \Longrightarrow \vec{F}$ is not cons.

Theorem Assume that $\vec{F}(x,y,z)$ has continuous partial derivatives and $\operatorname{curl} \vec{F} = \vec{0}$

$$\Longrightarrow \quad \vec{F} = \nabla f.$$

Theorem $\vec{F} = \langle P, Q, R \rangle$, P, Q, R have continuous second-order partial derivatives

$$\Longrightarrow \underline{\operatorname{div} \operatorname{curl} \vec{F} = 0}$$

Proof $\operatorname{div} \operatorname{curl} \vec{F} = \operatorname{div} \langle \frac{\partial R}{\partial y} -$

$$\vec{F} = \nabla f + \operatorname{curl} \vec{G}$$

Ex. 5 Show that $\vec{F} = \langle xz, xyz, -y^2 \rangle$ cannot be written as the curl of another vector field, that is, $\underline{\vec{F} \neq \operatorname{curl} \vec{G}}$

Assume that $\vec{F} = \operatorname{curl} \vec{G}$

$$\Rightarrow \operatorname{div} \vec{F} = \operatorname{div} \operatorname{curl} \vec{G} = 0$$

$$\operatorname{div} \vec{F} = \operatorname{div} \langle xz, xyz, -y^2 \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= z + xz + 0 = z + xz \neq 0$$

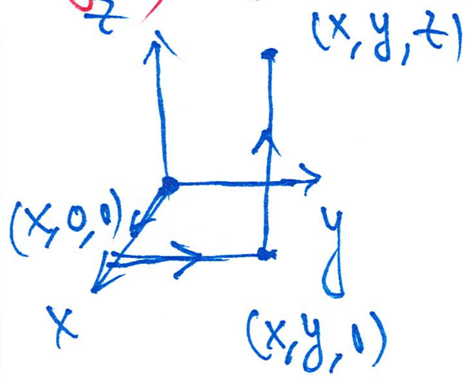
$$\Rightarrow \vec{F} \neq \operatorname{curl} \vec{G}$$

Ex. 3 (a) Show that $\vec{F} = \langle y^2 z^3,\ 2xyz^3,\ 3xy^2z^2 \rangle$ is a conservative vector field.

(b) Find a function f such that $\vec{F} = \nabla f$.

$$\nabla \times \vec{F} = \langle 6xyz^2 - 6xyz^2,\ 3y^2z^2 - 3y^2z^2,\ 2yz^3 - 2yz^3 \rangle = \langle 0,0,0 \rangle = \vec{0}$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 3xy^2z^2 \\ \dfrac{\partial f}{\partial y} = 2xyz^3 \\ \dfrac{\partial f}{\partial z} = y^2z^3 \end{cases}$$



$$f(x,y,z) - \underline{f(0,0,0)} = \int_C \nabla f \cdot d\vec{r}$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^x P(x,0,0)\,dx + \int_0^y Q(x,y,0)\,dy + \int_0^z R(x,y,z)\,dz$$

$$= \int_0^x 0 + \int_0^y 0 + \int_0^z 3xy^2z^2\,dz = xy^2z^3 \Big|_0^z = \underline{xy^2z^3}$$

- <u>divergence</u>

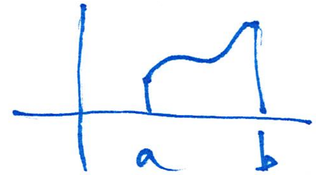
$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

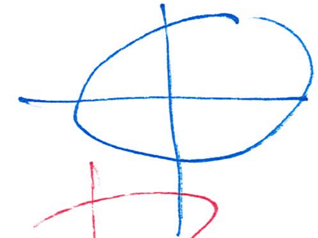
Ex. 4 $\vec{F} = \langle xz,\ xyz,\ -y^2 \rangle$

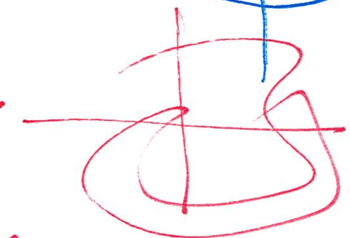
$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = z + xz + 0 = z + xz$$

§16.6 Parametric Surfaces and Their Areas

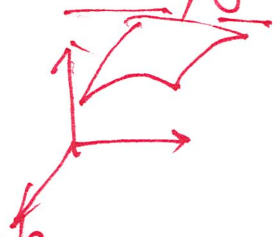
curves in R^2
- graph $y = f(x)$
- level curve $f(x,y) = k$
- parametric curve $\vec{r}(t) = \langle x(t), y(t) \rangle$



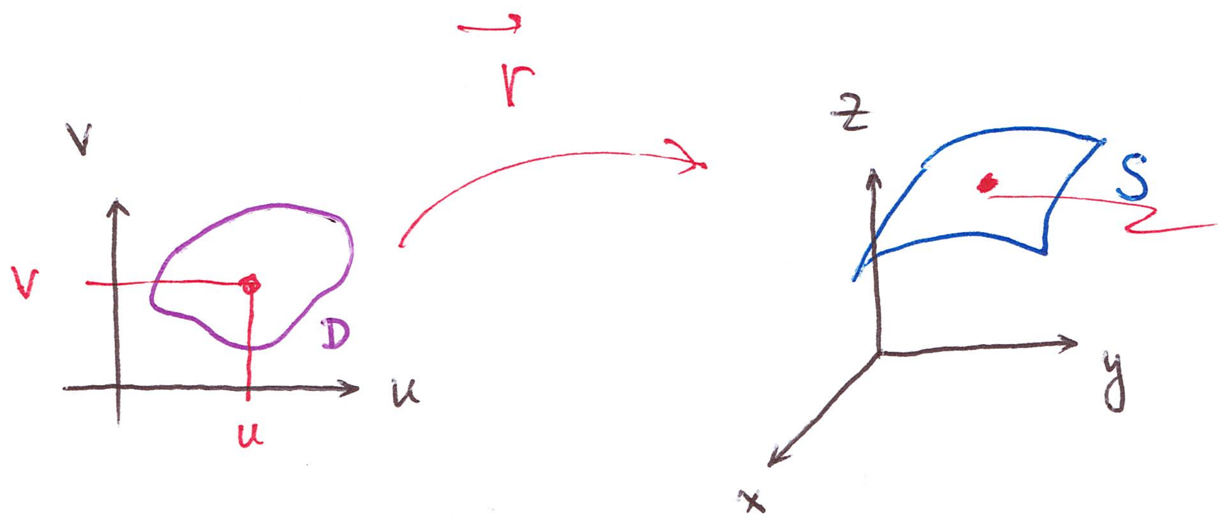




surfaces in R^3
- graph $z = f(x,y)$
- level surface $f(x,y,z) = k$
- parametric surface $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$



$(u,v) \in D$



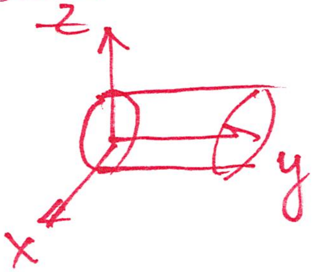
$\langle x(u,v), y(u,v), z(u,v) \rangle$

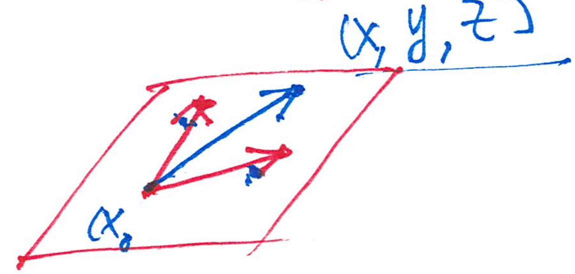
Ex. 1 Identify ~~the~~ and sketch the surface with vector equation

$$\vec{r}(u,v) = \langle 2\cos u,\ v,\ 2\sin u \rangle$$

$$\begin{cases} x = 2\cos u \\ y = v \\ z = 2\sin u \end{cases}$$

$$x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4$$



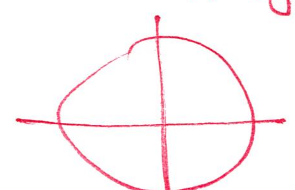


Examples Find parametric representations.

(1) the plane passing through three points (x_0, y_0, z_0), (x_1, y_1, z_1), and (x_2, y_2, z_2)

$$\langle x - x_0,\ y - y_0,\ z - z_0 \rangle = u \langle x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0 \rangle + v \langle x_2 - x_0,\ y_2 - y_0,\ z_2 - z_0 \rangle$$

$x^2 + y^2 = a^2$



$$\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$$

$0 \le t \le 2\pi$

(2) the sphere $x^2 + y^2 + z^2 = a^2$

$$\begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases} \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{array}$$