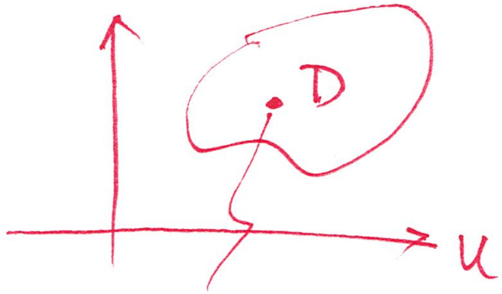
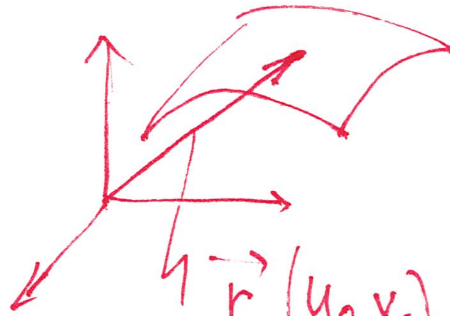
parametric surface

$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$

$(u,v) \in D$



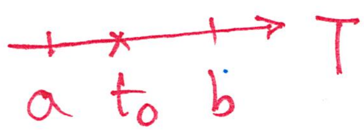
$\xrightarrow{\vec{r}}$



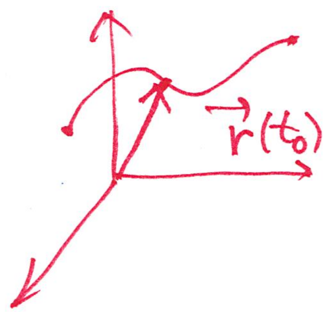
$\vec{r}(u_0, v_0) = \langle x(u_0, v_0),\ y(\ ,\ ),\ z(\ ,\ ) \rangle$

parametric curve

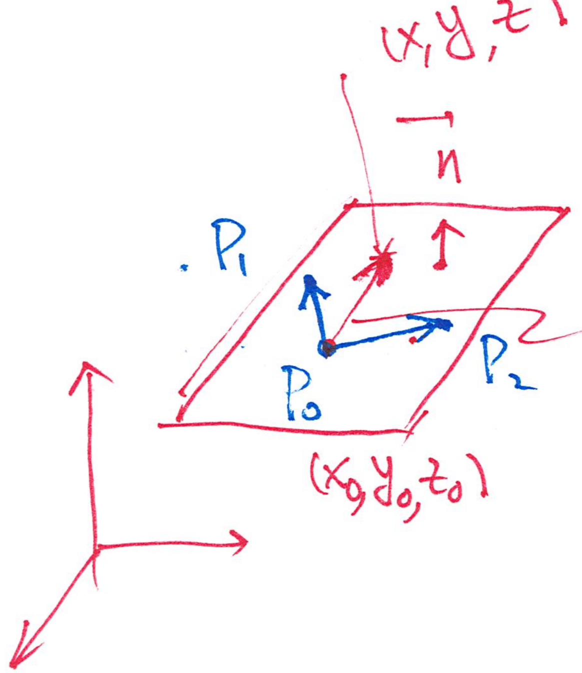
$\vec{r}(t) = \langle x(t),\ y(t),\ z(t) \rangle, \quad t \in [a, b]$



$\xrightarrow{\vec{r}}$



$\vec{r}(t_0) = \langle x(t_0),\ y(t_0),\ z(t_0) \rangle$



(1) $P_0(x_0, y_0, z_0)$ and $\underline{\vec{n} \perp P}$

$\langle x-x_0, y-y_0, z-z_0 \rangle$

$$0 = \vec{n} \cdot \langle x-x_0, y-y_0, z-z_0 \rangle$$

$$= n_1(x-x_0) + n_2(y-y_0) + n_3(z-z_0)$$

(2) $P_0(x_0, y_0, z_0)$, $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$

$$\underline{\langle x, y, z \rangle} - \langle x_0, y_0, z_0 \rangle \quad \langle x-x_0, y-y_0, z-z_0 \rangle = u\overrightarrow{P_0P_1} + v\overrightarrow{P_0P_2}$$

$$= u\langle x_1-x_0, y_1-y_0, z_1-z_0 \rangle + v\langle x_2-x_0, y_2-y_0, z_2-z_0 \rangle$$

$$\vec{r}(u,v) = \langle x_0 + u(x_1-x_0) + v(x_2-x_0), \; y_0 + u(y_1-y_0) + v(y_2-y_0), \; z_0 + u(z_1-z_0) + v(z_2-z_0) \rangle = \langle x(u,v), y(u,v), z(u,v) \rangle$$

(3) the cylinder $x^2+y^2=4$, $0 \le z \le 1$.

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = z \end{cases} \qquad \begin{matrix} 0 \le t \le 2\pi \\ 0 \le z \le 1 \end{matrix}$$

$\vec{r}(t,z) = \langle 2\cos t, 2\sin t, z \rangle$

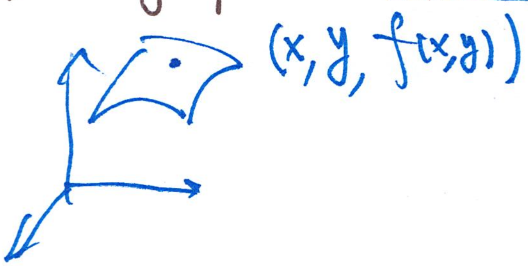
(4) the elliptic paraboloid $z = x^2+2y^2$.

$\vec{r}(u,v) = \langle u, v, u^2+2v^2 \rangle$

(5) the surface $z = 2\sqrt{x^2+y^2}$, that is, the top half of the cone $z^2 = 4x^2+4y^2$.

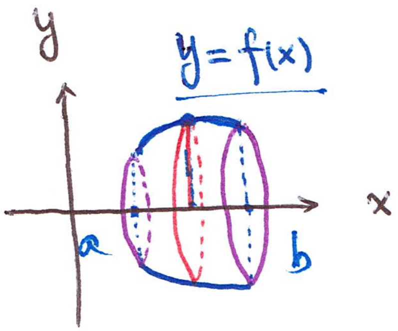
$\vec{r}(u,v) = \langle u, v, 2\sqrt{u^2+v^2} \rangle$

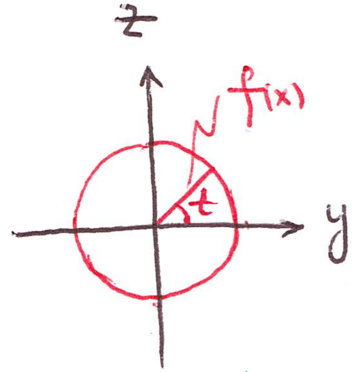
(6) the graph $z = f(x,y)$, $(x,y) \in D$. $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$, $(x,y) \in D$



$\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$, $(u,v) \in D$

- <u>surfaces of revolution</u>





parametrization

$$\begin{cases} x = x & a \leq x \leq b \\ y = f(x)\cos t & 0 \leq t \leq 2\pi \\ z = f(x)\sin t \end{cases}$$

<u>Ex. 8</u> Find parametric equations for the surface generated by rotating the curve $y=\sin x$, $0 \leq x \leq 2\pi$, about the x-axis.

$$\vec{r}(x,t) = \langle x, \sin x \cos t, \sin x \sin t \rangle$$

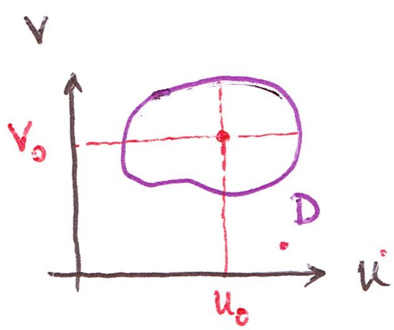
$0 \leq t \leq 2\pi$

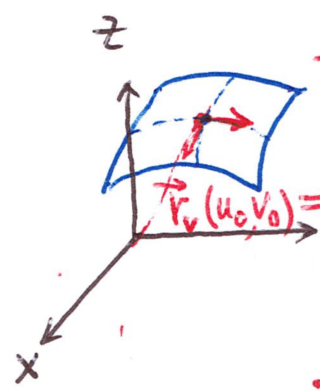
$0 \leq x \leq 2\pi$

- <u>surface</u> $S = \{\vec{r}(u,v) \mid (u,v) \in D\}$

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

(1) S is smooth at $\vec{r}(u_0,v_0) \iff \vec{r}_u \times \vec{r}_v (u_0,v_0) \neq 0$





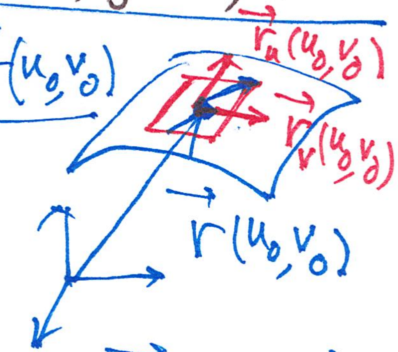
$$\vec{r}_u(u_0,v_0) = \left\langle \frac{\partial x(u,v)}{\partial u}, \frac{\partial y(u,v)}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \Bigg|_{(u_0,v_0)}$$

$$\vec{r}_v(u_0,v_0) = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \Bigg|_{(u_0,v_0)}$$

(2) tangent plane at $\vec{r}(u_0,v_0)$ $= \langle x(u_0,v_0), y(u_0,v_0), z(u_0,v_0)\rangle$ to surface $r(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

- parametric representation $\vec{r}(u,v) = u\,\vec{r}_u(u_0,v_0) + v\,\vec{r}_v(u_0,v_0) + \vec{r}(u_0,v_0)$

- graph representation $0 = \left(\vec{r}(u,v) - \vec{r}(u_0,v_0)\right)\cdot\vec{n}$



$\vec{n} = \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$

Ex. 9 Find the tangent plane to the surface $\vec{r}(u,v) = \langle u^2, v^2, u+2v\rangle$ at $(1,1,3)$.

$\vec{r}_u = \langle 2u, 0, 1\rangle$ $\qquad\qquad$ $\vec{r}_u(1,1) = \langle 2, 0, 1\rangle$

$\vec{r}_v = \langle 0, 2v, 2\rangle$ $\qquad\qquad$ $\vec{r}_v(1,1) = \langle 0, 2, 2\rangle$

parametric $\quad \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle = \langle 1, 1, 3\rangle + u\langle 2,0,1\rangle + v\langle 0,2,2\rangle$

$= \langle 1+2u, 1+2v, 3+u+2v\rangle$

graph $\quad \vec{n} = \begin{vmatrix} i & j & k \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = \langle -2, -4, 4\rangle = -2\underbrace{\langle 1, 2, -2\rangle}$

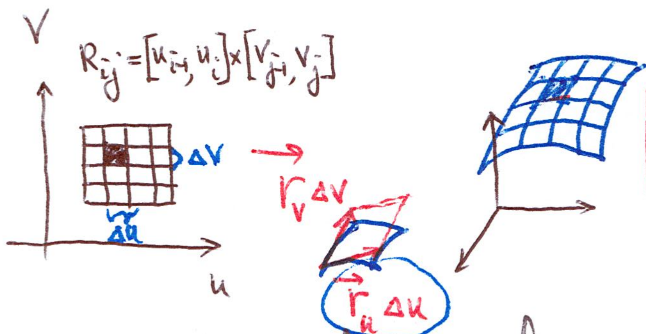
$0 = \langle 1, 2, -2\rangle \cdot \langle x-1, y-1, z-3\rangle$

$= (x-1) + 2(y-1) - 2(z-3)$

(3) surface area

• parametric surface $S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, $(u,v) \in D$

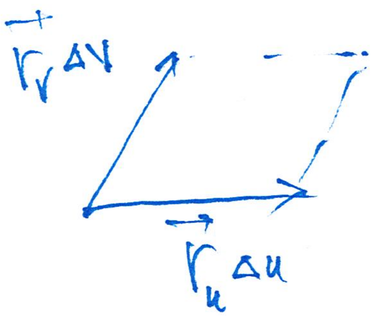
is a smooth surface and is covered once



$$\boxed{A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA}$$

$$dS = |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

Ex. 10 Find the surface area of a sphere of radius a.



$$|\vec{r}_u \Delta u \times \vec{r}_v \Delta v| = |\vec{r}_u \times \vec{r}_v| \, \Delta u \Delta v$$

$$\begin{cases} x = a \sin\varphi \cos\theta & 0 \le \varphi \le \pi \\ y = a \sin\varphi \sin\theta & 0 \le \theta \le 2\pi \\ z = a \cos\varphi \end{cases}$$

$$\vec{r}(\theta,\varphi) = \langle a \sin\varphi \cos\theta, a \sin\varphi \sin\theta, a \cos\varphi \rangle$$

$$\begin{array}{cccc} & i & j & k \\ \vec{r}_\theta = & \langle -a \sin\varphi \sin\theta, & a \sin\varphi \cos\theta, & 0 \rangle \\ \vec{r}_\varphi = & \langle a \cos\varphi \cos\theta, & a \cos\varphi \sin\theta, & -a \sin\varphi \rangle \end{array}$$

$$\vec{r}_\theta \times \vec{r}_\varphi = a^2 \langle -\sin^2\varphi \cos\theta, -\sin^2\varphi \sin\theta, -\sin\varphi \cos\varphi \rangle$$

$$= -a \sin\varphi \, \boxed{a \langle \sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi \rangle}$$

$$\boxed{\vec{r}_\theta \times \vec{r}_\varphi = -a \sin\varphi \, \vec{r}(\theta,\varphi)}$$

$$|\vec{r}_\theta \times \vec{r}_\varphi| = +a \sin\varphi \, |\vec{r}(\theta,\varphi)| = +a \sin\varphi \cancel{\sqrt{\sin^2\varphi + \cos^2\varphi}}$$

$$A(S) = \iint_D a^2 \sin\varphi \, dA = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\varphi \, d\varphi \, d\theta$$

$$= a^2 \cdot 2\pi \int_0^{\pi} \sin\varphi \, d\varphi = 2\pi a^2 \left[-\cos\varphi \right]_0^{\pi}$$

$$= 4\pi a^2$$

• graph $S: z = f(x,y),\ (x,y) \in D \implies \vec{r}(u,v) = \langle u, v, f(u,v) \rangle$

$(u,v) \in D$

$$A(S) = \iint_D \underbrace{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}_{\parallel \atop |\vec{r}_x \times \vec{r}_y|}\, dA$$

$\vec{r}_u = \langle 1, 0, \frac{\partial f}{\partial u} \rangle$

$\vec{r}_v = \langle 0, 1, \frac{\partial f}{\partial v} \rangle$

$\vec{r}_u \times \vec{r}_v = \langle -\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1 \rangle$

Ex. 11 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.